Mathematical World • Volume 8

Mathematics 1

Japanese Grade 10

Kunihiko Kodaira, Editor

American Mathematical Society
The University of Chicago School Mathematics Project

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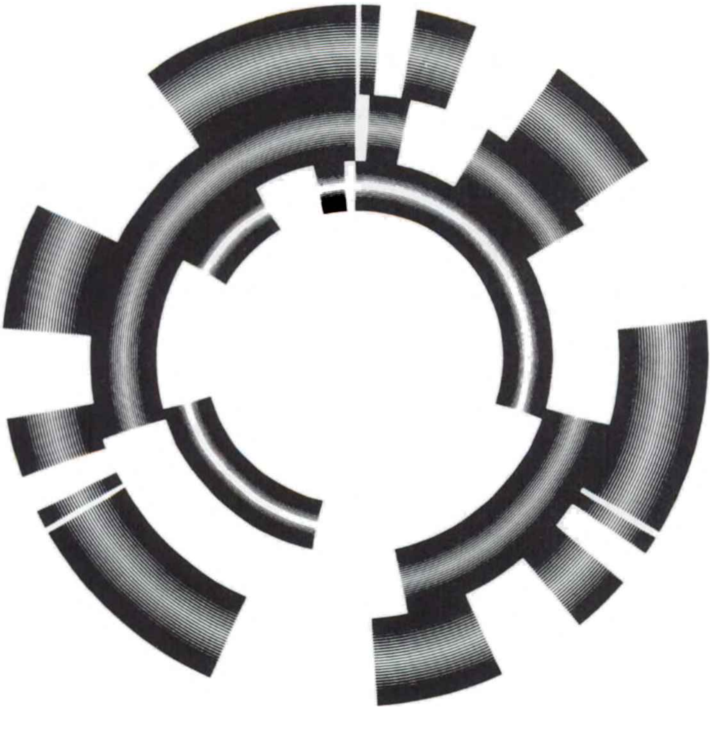
昭和56年 3 月31日文部省検定済
高等学校数学科用

数学 I

小平邦彦 編



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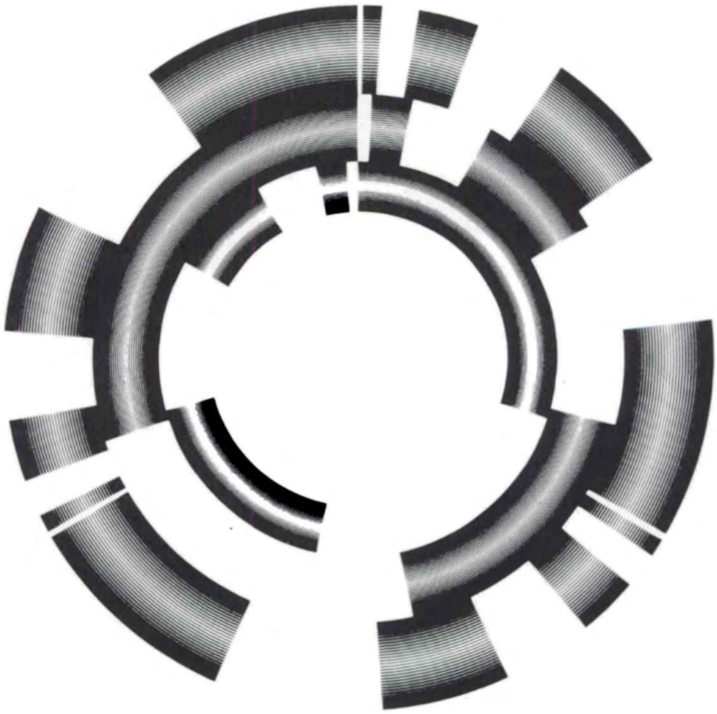
Mathematics 1
Japanese Grade 10



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The University of Chicago School Mathematics Project
Zalman Usiskin, Director
Izaak Wirszup, Director, Resource Development Component

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Translated by Hiromi Nagata
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Textbook Series Preface

The University of Chicago School Mathematics Project

This textbook is part of a series of foreign mathematics texts that have been translated by the Resource Development Component of the University of Chicago School Mathematics Project (UCSMP). Established in 1983 with major funding from the Amoco Foundation, UCSMP has been since that time the nation's largest curriculum development and implementation project in school mathematics. The international focus of its resource component, together with the project's publication experience, makes UCSMP well suited to disseminate these remarkable materials.

The textbooks were originally translated to give U.S. educators and researchers a first-hand look at the content of mathematics instruction in educationally advanced countries. More specifically, they provided input for UCSMP as it developed new instructional strategies, textbooks, and materials of its own; the resource component's translations of over 40 outstanding foreign school mathematics publications, including texts, workbooks, and teacher aids, have been used in UCSMP–related research and experimentation and in the creation of innovative textbooks.

The resource component's translations include the entire mathematics curriculum (grades 1–10) used in the former Soviet Union, standard Japanese texts for grades 7–11, and innovative elementary textbooks from Hungary and Bulgaria.

The content of Japan's compulsory national curriculum for grades 7–11 is made available for the first time in English, thanks in part to the generosity of the Japanese publisher, Tokyo Shoseki Company, Ltd., which provided the copyright permissions.

Japanese Secondary School Mathematics Textbooks

The achievement of Japanese elementary and secondary students gained world prominence largely as a result of their superb performance in the International Mathematics Studies conducted by the International Association for the Evaluation of Educational Achievement. The Second International Mathematics Study surveyed mathematics achievement in 24 countries in 1981–82 and released its findings in 1984. The results are recapitulated in a 1987 national report entitled "The Underachieving Curriculum: Assessing U.S. School Mathematics from an International Perspective" (A National Report on the Second International Mathematics Study, 1987).

Let us take a brief look at the schooling behind much of Japan's economic success. The Japanese school system consists of a six-year primary school, a three-year lower secondary school, and a three-year upper secondary school. The first nine grades are compulsory, and enrollment is now 99.99%. According to 1990 statistics, 95.1% of age-group children are enrolled in upper secondary school, and the dropout rate is 2.2%. In terms of achievement, a typical Japanese student graduates from secondary school with roughly four more years of education than an average American high school graduate. The level of mathematics training achieved by Japanese students can be inferred from the following data:

Japanese Grade 7 Mathematics (New Mathematics 1) explores integers, positive and negative numbers, letters and expressions, equations, functions and proportions, plane figures, and figures in space. Chapter headings in *Japanese Grade 8 Mathematics* include calculating expressions, inequalities, systems of equations, linear functions, parallel lines and congruent figures, parallelograms, similar figures, and organizing data. *Japanese Grade 9 Mathematics* covers square roots, polynomials, quadratic equations, functions, circles, figures and measurement, and probability and statistics. The material in these three grades (lower secondary school) is compulsory for all students.

The textbook *Japanese Grade 10 Mathematics 1* covers material that is compulsory. This course, which is completed by over 97% of all Japanese students, is taught four hours per week and comprises algebra (including quadratic functions, equations, and inequalities), trigonometric functions, and coordinate geometry.

Japanese Grade 11 General Mathematics 2 is intended for the easier of the electives offered in that grade and is taken by about 40% of the students. It covers probability and statistics; vectors; exponential, logarithmic, and trigonometric functions; and differentiation and integration in an informal presentation.

The other 60% of students in grade 11 concurrently take two more extensive courses using the texts *Japanese Grade 11 Algebra and Geometry* and *Japanese Grade 11 Basic Analysis*. The first consists of fuller treatments of plane and solid coordinate geometry, vectors, and matrices. The second includes a more thorough treatment of trigonometry and an informal but quite extensive introduction to differential and integral calculus.

Some 25% of Japanese students continue with mathematics in grade 12. These students take an advanced course using the text *Probability and Statistics* and a second rigorous course with the text *Differential and Integral Calculus*.

One of the authors of these textbooks is Professor Hiroshi Fujita, who spoke at UCSMP's International Conferences on Mathematics Education in 1985, 1988, and 1991. Professor Fujita's paper on Japanese mathematics education appeared in *Developments in School Mathematics Education Around the World*, volume 1 (NCTM, 1987). The current school mathematics reform in Japan is described in the article "The Reform of Mathematics Education at the Upper Secondary School (USS) Level in Japan" by Professors Fujita, Tatsuro Miwa, and Jerry Becker in the proceedings of the Second International Conference, volume 2 of *Developments*.

Acknowledgments

It goes without saying that a publication project of this scope requires the commitment and cooperation of a broad network of institutions and individuals. In acknowledging their contributions, we would like first of all to express our deep appreciation to the Amoco Foundation. Without the Amoco Foundation's generous long-term support of the University of Chicago School Mathematics Project these books might never have been translated for use by the mathematics education community.

We are grateful to UCSMP Director Zalman Usiskin for his help and advice in making these valuable resources available to a wide audience at low cost. Robert Streit, Manager of the Resource Development Component, did an outstanding job coordinating the translation work and collaborating on the editing of most of the manuscripts. George Fowler, Steven R. Young, and Carolyn J. Ayers made a meticulous review of the translations, while Susan Chang and her technical staff at UCSMP handled the text entry and layout with great care and skill. We gratefully acknowledge the dedicated efforts of the translators and editors whose names appear on the title pages of these textbooks.

Izaak Wirszup, Director
UCSMP Resource Development Component

FOREWORD TO THE JAPANESE EDITION

The history of mathematics goes back thousands of years, and the discipline has undergone continuous advances. In the last few decades, particularly, mathematics has made startling progress.

Geometry is said to have started in ancient Egypt and Babylonia with astronomical observations, measurement of land, and architecture. Thus, the advancement of mathematics was originally linked with science and technology; however, it gradually became independent of science and technology, and present-day mathematicians think freely about virtually everything possible. Therefore, mathematics is said to be a free creation of the human spirit.

This mathematics which is created freely by the human spirit can nonetheless be very useful when it is applied to various areas of study. For example, complex numbers were introduced as a way to imagine non-existing entities; this fact is reflected in the term "imaginary number." But present-day physics would be unthinkable without complex numbers. The realm of complex numbers is the real, existing world, and the realm of real numbers is now thought to be only a small part of that world. In this respect, even if mathematics can be said to be a free creation of the human spirit, it is not created freely by human beings. In other words, there exists a world of mathematical phenomena, just as physics is the study of natural phenomena, and mathematics is the science that studies those mathematical phenomena. Moreover, since the world of mathematical phenomena underlies the natural world, we must realize that mathematics is useful for the natural sciences. Thus, mathematics is a critical discipline; it is basic for the study of various other sciences.

This book is a textbook for students who have graduated from junior high school and entered high school to study Mathematics I. In Chapter 1 you will continue to learn basic concepts involving numbers, expressions, and fractional expressions, etc. which you also studied in junior high school. In Chapter 2 you will learn about equations and inequalities, especially a method of solving quadratic equations, and about complex numbers in connection with this method. You have already studied quadratic equations in junior high school, but in this chapter you will study quadratic equations from a broader standpoint and you will reach complex numbers. Figures are objects with completely different properties from numbers and polynomials; however, there is a close relationship between figures and polynomials. In Chapter 3 you will learn about the relation between figures and polynomials. Chapter 4 presents functions and their graphs. The fifth and last chapter deals with trigonometry and the relation between the properties of triangles and trigonometry.

To master mathematics, it is not enough to read and memorize books. It is important to think through the material, do calculations, and solve problems by yourselves. In this book we try to explain the material in a simple way by using many examples and sample problems.

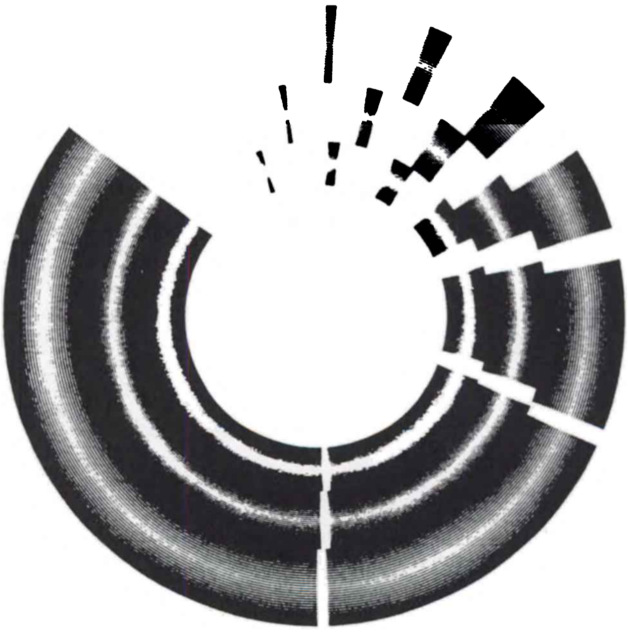
TABLE OF CONTENTS

To the Student

Example	This marker designates a concrete example to help you understand the text.
Demonstration	This heading precedes a standard problem for better understanding of the material. Boxes labeled **[Solution]** and **[Proof]** give model answers.
Note:	This marker indicates an explanation to help you understand a particular point.
Problem 1	Problems for rapid mastery of current material and for introducing new material appear in the text with this label.
Exercises	At the end of each section problems are provided for practice with the material in that section.
Chapter Exercises	At the end of each chapter problems are provided for review of the entire chapter and practical application of the material. A problems involve primarily basic material, while B problems are a bit more advanced.
Reference	Refers to topics from other elective courses.

CHAPTER 1

NUMBERS AND EXPRESSIONS



The concept of real numbers can be understood intuitively by relating them to the points on the number line. This concept received a formal mathematical definition only in the second half of the nineteenth century. The concept of real numbers provides a basic demonstration of the efficiency of the concept of a set. "The continuity of real numbers" arises both explicitly and implicitly in various areas of science, and the practical usefulness of complex numbers, introduced in Chapter 2, stems from the continuity of real numbers. Extending the set of numbers to encompass real numbers by adding irrational numbers to rational numbers has other uses besides the mere calculation of square roots.

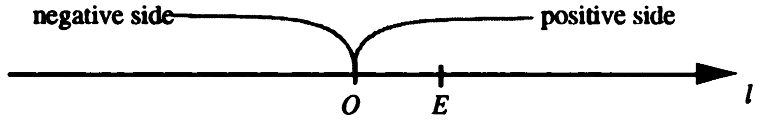
An algebraic symbolic system that represents both unknown and known quantities as letters was first introduced by Vieta (1540–1603) and perfected by Descartes (1596–1650). This approach systematized algebra and laid a foundation for the extensive development of algebra up to the present day.

1 REAL NUMBERS

1 The Number Line and Real Numbers

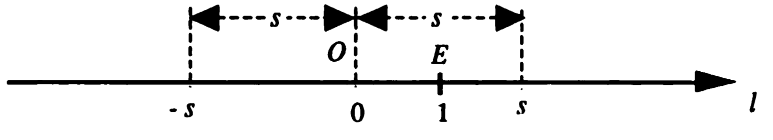
Let's consider numbers and the number line again, as well as review what you have already learned through junior high school.

Point O and another point E are given on a single straight line l. Excluding point O, line l is divided into two parts on either side of point O. The part that contains point E is called the positive side, and the part without point E is called the negative side.



The points on line l correspond to numbers in the following way.

Point O corresponds to the number 0. For any point on the positive or negative side, let s be its distance from point O, taking segment OE as the unit of length. When the point is on the positive side, it corresponds to the positive number s, and when it is on the negative side, it corresponds to the negative number $-s$.



In this way, every point on l corresponds to a unique number: different points correspond to different numbers. The number that corresponds to point P is called the **coordinate** of point P, and the notation $P(x)$ signifies that x is the coordinate of point P. Point P is sometimes referred to as the point x for short.

When each point on line l is indicated by the number which is its coordinate, l is called the **number line**, O is the **origin**, and E is the **unit point**. The coordinate of the origin is 0, and the coordinate of the unit point is 1. Numbers expressed as points on the number line are called **real numbers**.

When the number line is drawn horizontally, the positive side is usually to the right and the negative side is to the left.

Problem 1 Draw a number line, and indicate the following points on it:

(1) 4 (2) -3 (3) 2.5 (4) -1.5 (5) $-\frac{3}{4}$

Comparison of Real Numbers

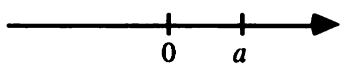
On the horizontal number line, positive numbers are indicated as points to the right of the origin and negative numbers are indicated as points to the left of the origin. When we say that real number b is **greater than** real number a, or that a is **less than** b, we mean that point b is to the right of point a. This relation is represented by the notations

$$a < b \text{ or } b > a.$$

A combination whereby either $a < b$ or $a = b$ is represented by $a \leq b$.

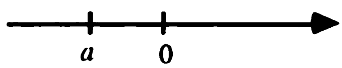
If a is positive, we can write

$$a > 0,$$



and if a is negative, we write

$$a < 0.$$



The following properties apply to the comparison of real numbers.

(I) For any real numbers a and b, one of the following three relations holds:

$$a < b, \ a = b, \ a > b.$$

(II) $a < b, \ b < c \Rightarrow a < c$

Note: The symbol $\Rightarrow$ means "implies."

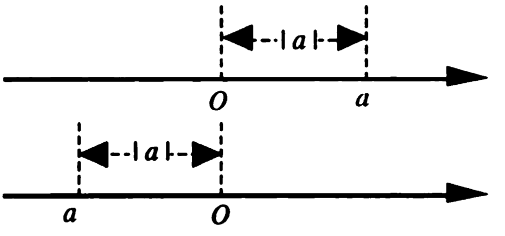
Absolute Values of Real Numbers

The distance from the origin to point a on the number line is called the absolute value of the number a.

The absolute value of 3 is 3, and the **absolute value** of -3 is also 3.

The absolute value of 0 is 0.

The absolute value of the real number a is denoted by $|a|$.



Example $|3| = 3$, $|-3| = 3$, $|0| = 0$.

Problem 2 Give the following values:

(1) $|8|$ (2) $|-10|$ (3) $|2.5|$ (4) $\left|-\frac{1}{3}\right|$ (5) $\left|\frac{5}{6}\right|$

Problem 3 Give the real number x which satisfies $|x| = 7$.

From the definition of an absolute value, the following properties hold:

(1)	$a \geq 0 \Rightarrow |a| = a$.
(2)	$a \leq 0 \Rightarrow |a| = -a$.
(3)	For any real number a, $|a| \geq 0$.
(4)	$|a| = 0$ is true if and only if $a = 0$.

Demonstration Prove that the following equation is true for any real number a:

$$|-a| = |a|.$$

[Proof] Consider this equation for $a \geq 0$ and $a < 0$.

(i) For $a \geq 0$, since $-a \leq 0$,

$$|-a| = -(-a) = a, \ |a| = a.$$

(ii) For $a < 0$, since $-a > 0$,

$$|-a| = -a, \ |a| = -a.$$

By (i) and (ii), for any real number a

$$|-a| = |a|.$$

Problem 4 Prove that $a^2 = |a|^2$ is true for any real number a.

1 Calculations Involving Real Numbers

The sum, difference, and product of any real numbers a and b are all real numbers. If $b \neq 0$, then the quotient $\frac{a}{b}$ is also a real number.

The following laws apply to addition and multiplication of real numbers.

Commutative law: $a + b = b + a$, $ab = ba$.

Associative law: $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$.

Distributive law: $a(b + c) = ab + ac$, $(a + b)c = ac + bc$.

Since $a - b$ is equal to $a + (-b)$ and $\frac{a}{b}$ is equal to $a \times \frac{1}{b}$, subtraction and division can be treated as addition and multiplication.

Relations with Inequalities

Properties I – III below relate inequalities for calculations involving real numbers.

(I) For any real number c,

$$a > b \Rightarrow a + c > b + c.$$

(II) For $c > 0$,

$$a > b \Rightarrow ac > bc, \ \frac{a}{c} > \frac{b}{c}.$$

(III) For $c < 0$,

$$a > b \Rightarrow ac < bc, \ \frac{a}{c} < \frac{b}{c}.$$

Example For $a > 0$ and $b > 0$, if we add b to both sides of $a > 0$, then

by (I), $a + b > b$,

since $b > 0$, $a + b > 0$.

Problem 1 Prove that $a + b < 0$ for $a < 0$ and $b < 0$, as in the example above.

The Sign of a Sum

(1) $a > 0, b > 0 \Rightarrow a + b > 0.$

(2) $a < 0, b < 0 \Rightarrow a + b < 0.$

When a and b have different signs, the sign of $a + b$ depends on the relative absolute values of a and b.

The Sign of a Product or a Quotient

(1) $a > 0,\ b > 0 \Rightarrow ab > 0,\ \frac{a}{b} > 0.$

(2) $a > 0,\ b < 0 \Rightarrow ab < 0,\ \frac{a}{b} < 0.$

(3) $a < 0,\ b > 0 \Rightarrow ab < 0,\ \frac{a}{b} < 0.$

(4) $a < 0,\ b < 0 \Rightarrow ab > 0,\ \frac{a}{b} > 0.$

[Proof] (1) For $a > 0$ and $b > 0$, property (II) above tells us that if we multiply both sides of $a > 0$ by b, then

$$ab > 0.$$

If we divide both sides of $a > 0$ by b, then

$$\frac{a}{b} > 0.$$

Problem 2 Prove properties (2) – (4) of the sign of a product or a quotient.

Problem 3 Prove the following implications using properties (1) and (2) of the sign of a product or a quotient.

(1) $a > 0 \Rightarrow \frac{1}{a} > 0,$ (2) $a < 0 \Rightarrow \frac{1}{a} < 0.$

Problem 4 Prove that if a and b are positive numbers, $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$.

By properties (1) and (4) of the sign of a product or a quotient,

$$a > 0 \Rightarrow a^2 > 0,$$

$$a < 0 \Rightarrow a^2 > 0.$$

In other words,

$$a \neq 0 \Rightarrow a^2 > 0.$$

If we include $a = 0$, the following generalization holds.

> For any real number a, $a^2 \geq 0$.
>
> $a^2 = 0$ is true if and only if $a = 0$.

Demonstration Prove that for real numbers a and b, $a^2 + b^2 = 0$ implies $a = b = 0$.

[Proof] Since $a^2 \geq 0$ and $b^2 \geq 0$,

$$a^2 + b^2 \geq a^2 \geq 0.$$

If $a^2 + b^2 = 0$, then

$$0 \geq a^2 \geq 0 \text{ or } a^2 = 0.$$

Therefore, $a = 0$.

The proof of $b = 0$ can be obtained in the same way.

Problem 5 Prove that for real numbers a and b, $(a-1)^2 + (b-2)^2 = 0$ implies $a = 1$ and $b = 2$.

If we add $-b$ to both sides of $a > b$, we obtain $a - b > 0$. Therefore,

$$a > b \Rightarrow a - b > 0.$$

Conversely, if we add b to both sides of $a - b > 0$, we obtain $a > b$. So

$$a - b > 0 \Rightarrow a > b.$$

We can combine these two facts in the following way:

$$\boxed{a > b \Leftrightarrow a - b > 0.}$$

This property is often used for comparing real numbers.

Note: The notation $p \Leftrightarrow q$ means that "$p \Rightarrow q$ and $q \Rightarrow p$."

Problem 6 Prove that $a - b < 0 \Leftrightarrow a < b$.

3 Classification of Real Numbers

The numbers we obtain if we start with 1 and then add 1 over and over,

$$1, 2, 3, \ldots$$

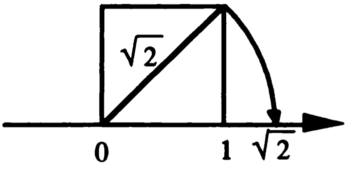
are called **natural numbers** or **positive integers**. Sums and products of natural numbers are also natural numbers.

$$-1, -2, -3, \ldots$$

are called **negative integers**. Positive integers, negative integers, and 0 make up the class of **integers**. Sums, differences, and products of integers are also integers.

Numbers of the form $\frac{m}{n}$, where m and n are integers, are called **rational numbers**. Since any integer m can be expressed in the form $\frac{m}{1}$, it is therefore a rational number. Sums, differences, products, and quotients of rational numbers are also rational numbers.

There are many points on the number line which do not express rational numbers. For example, the length of the diagonal of a square with a side of 1 equal to $\sqrt{2}$ can be expressed as a point on the number line; however, we know that it is not a rational number.



Real numbers which are not rational are called **irrational numbers**. $\sqrt{2}$ is an irrational number. The circular constant π is also an irrational number.

Classification of Real Numbers

- Real numbers
  - Rational numbers
    - Integers
      - Positive integers: 1, 2, 3, ...
      - 0
      - Negative integers: -1, -2, -3, ...
    - Non-integer rational numbers: $\frac{1}{8}$, $-\frac{4}{3}$, ...
  - Irrational numbers: $\sqrt{2}$, $-\sqrt{3}$, π, ...

Non-integer rational numbers may be represented as finite decimals, such as

$$\frac{1}{8} = 0.125$$

or they may be expressed as infinite decimals with a repeating sequence of digits after the decimal place, such as

$$-\frac{4}{3} = -1.3333...$$

$$\frac{9}{74} = 0.1216216216...$$

This kind of infinite decimal is called a **repeating decimal**.
Irrational numbers are expressed as non-repeating infinite decimals, for example:

$$\sqrt{2} = 1.41421...$$

$$\pi = 3.14159...$$

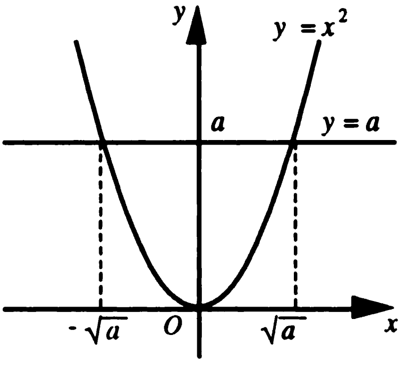
Problem Identify the natural numbers, integers, rational numbers, and irrational numbers among the following numbers:

$$0,\ 3,\ \pi,\ -5,\ \frac{2}{7},\ -\frac{9}{4},\ \sqrt{8},\ \sqrt{0.16},\ \sqrt{3}+1$$

3 Calculating Expressions Containing Radicals

A real number x is said to be the **square root** of a if the square of x is equal to a, or in other words, if it satisfies the following equality:

$$x^2 = a.$$



A positive number a has two square roots; their absolute values are the same, but their signs are different. The positive square root is designated by $\sqrt{a}$, and the negative square root is designated by $-\sqrt{a}$.

For example, the square roots of 4 are 2 and - 2, and $\sqrt{4}$ designates 2.

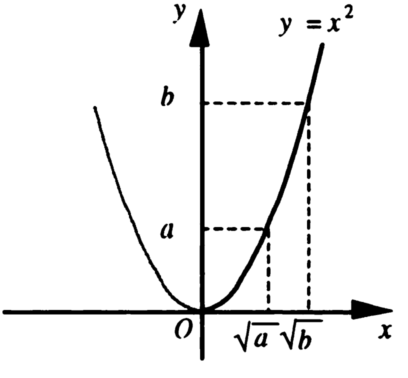
The square root of 0 is 0, and we define $\sqrt{0} = 0$.

Negative numbers have no square roots in the realm of real numbers.

Problem 1 Give an example of a real number a which does not satisfy the equality $\sqrt{a^2} = a$. Then prove that $\sqrt{a^2} = |a|$ is true for any real number a.

It is clear from the figure to the right that for any positive numbers a and b,

$$a < b \Leftrightarrow \sqrt{a} < \sqrt{b}.$$



The following formulas hold for square roots, as you learned in junior high school.

The Product and Quotient of Square Roots

For $a > 0$ and $b > 0$,

(1) $\sqrt{ab} = \sqrt{a}\sqrt{b}$, (2) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

Example 1 $\sqrt{20} = \sqrt{2^2 \times 5} = \sqrt{2^2}\sqrt{5} = 2\sqrt{5}$

$\sqrt{0.03} = \sqrt{\frac{3}{100}} = \frac{\sqrt{3}}{\sqrt{100}} = \frac{\sqrt{3}}{10}$

$\sqrt{10}\sqrt{45} = \sqrt{2 \times 5}\sqrt{3^2 \times 5} = \sqrt{3^2 \times 5^2 \times 2} = 15\sqrt{2}$

Problem 2 Transform the following expressions as in the preceding example:

(1) $\sqrt{32}$ (2) $\sqrt{108}$ (3) $\sqrt{\frac{75}{16}}$ (4) $\sqrt{0.09}$

Demonstration 1 Perform the following calculations:

(1) $\sqrt{12} - \sqrt{27} + \sqrt{\frac{3}{4}}$

(2) $(2\sqrt{2} + 5\sqrt{3})(3\sqrt{2} - \sqrt{3})$

[Solution] (1) $\sqrt{12} - \sqrt{27} + \sqrt{\frac{3}{4}} = \sqrt{2^2 \times 3} - \sqrt{3^2 \times 3} + \sqrt{\frac{3}{2^2}}$

$= 2\sqrt{3} - 3\sqrt{3} + \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$

(2) $(2\sqrt{2} + 5\sqrt{3})(3\sqrt{2} - \sqrt{3})$

$= (2\sqrt{2} + 5\sqrt{3}) \times 3\sqrt{2} - (2\sqrt{2} + 5\sqrt{3}) \times \sqrt{3}$

$= 12 + 15\sqrt{6} - 2\sqrt{6} - 15 = -3 + 13\sqrt{6}$

Problem 3 Perform the following calculations:

(1) $\sqrt{8}+\sqrt{18}-\sqrt{72}$ (2) $\sqrt{20}-3\sqrt{2}-\sqrt{\frac{5}{9}}+\sqrt{50}$

(3) $(\sqrt{3}-\sqrt{2})^2$ (4) $(3\sqrt{2}+\sqrt{3})(2\sqrt{3}-\sqrt{2})$

Demonstration 2 Compare the values of $1+\sqrt{3}$ and $\sqrt{7}$.

[Solution] Since both numbers are positive, we can compare their values by squaring both numbers.

$$\begin{aligned}(1+\sqrt{3})^2-(\sqrt{7})^2 &= (4+2\sqrt{3})-7\\ &= 2\sqrt{3}-3\\ &= \sqrt{12}-\sqrt{9}>0\end{aligned}$$

Therefore, $(1+\sqrt{3})^2 > (\sqrt{7})^2$.

Thus $1+\sqrt{3} > \sqrt{7}$.

Problem 4 Compare the values of each pair of numbers:

(1) $2\sqrt{7}$, $3\sqrt{3}$ (2) $2+\sqrt{6}$, $2\sqrt{5}$

Rationalizing the Denominator

Fractions whose denominators contain radicals can be transformed into fractions without radicals in their denominators in the following way.

Example 2 $\dfrac{1}{\sqrt{12}}=\dfrac{1}{2\sqrt{3}}=\dfrac{\sqrt{3}}{2(\sqrt{3})^2}=\dfrac{\sqrt{3}}{6}$

Example 3 $$\frac{\sqrt{5}}{\sqrt{5}-\sqrt{2}} = \frac{\sqrt{5}(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$$

$$= \frac{(\sqrt{5})^2+\sqrt{5}\sqrt{2}}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{5+\sqrt{10}}{3}$$

This kind of transformation is called **rationalizing the denominator.**

Problem 5 Rationalize the denominators in each of the following expressions:

(1) $\frac{1}{\sqrt{28}}$ (2) $\frac{\sqrt{3}}{\sqrt{3}+\sqrt{2}}$ (3) $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-\sqrt{5}}$

Problem 6 Find the approximate value of $\frac{3}{\sqrt{7}-2}$, taking 2.6458 as the approximate value of $\sqrt{7}$.

Simplifying Double Radicals

Since

$$(\sqrt{3}+\sqrt{2})^2 = 3+2\sqrt{3\times 2}+2 = 5+2\sqrt{6},$$

$\sqrt{5+2\sqrt{6}}$ can be transformed in the following way:

$$\sqrt{5+2\sqrt{6}} = \sqrt{3}+\sqrt{2}.$$

This kind of transformation is called **removing the double radical.**

Example 4 $\sqrt{7+2\sqrt{10}} = \sqrt{5+2\sqrt{5\times 2}+2} = \sqrt{5}+\sqrt{2}$

Example 5 $\sqrt{8-2\sqrt{15}} = \sqrt{5-2\sqrt{5\times 3}+3} = \sqrt{(\sqrt{5}-\sqrt{3})^2}$

Since $\sqrt{5}-\sqrt{3} > 0$,

$$\sqrt{8-2\sqrt{15}} = \sqrt{5}-\sqrt{3}.$$

Problem 7 Remove the double radical and simplify each expression:

(1) $\sqrt{4+2\sqrt{3}}$ (2) $\sqrt{6-2\sqrt{8}}$ (3) $\sqrt{7+\sqrt{24}}$

Exercises

1. Fill in the blanks with the appropriate inequality sign:

   For a real number x,

$$-3 < x < 3 \Leftrightarrow |x| \boxed{\phantom{<}}\ 3,$$

$$x < -3 \text{ or } 3 < x \Leftrightarrow |x| \boxed{\phantom{<}}\ 3.$$

2. Given that a and b are non-zero real numbers, which of the quantities A, B, C, and D below are equal when a and b have the same sign? Which are equal when a and b have different signs?

$$A = ab,\ B = |ab|,\ C = |a||b|,\ D = -ab$$

3. Express the range of values of the real number x using inequality signs for cases (1) and (2).

   (1) x has two digits in the integer part when it is expressed as a decimal.

   (2) When x is rounded off to one decimal place, its value is 3.4 .

4. Perform the following calculations:

   (1) $\sqrt{48} - \dfrac{\sqrt{27}}{2} + \dfrac{1}{\sqrt{12}}$ (2) $3\left(\dfrac{4-\sqrt{7}}{3}\right)^2 - 8\left(\dfrac{4-\sqrt{7}}{3}\right)$

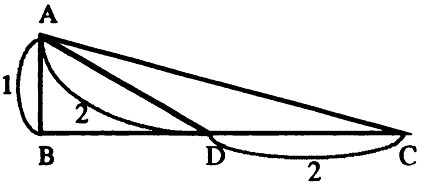
5. Find the value of the following expressions for $x = \dfrac{1}{\sqrt{7} - \sqrt{5}}$ and $y = \dfrac{1}{\sqrt{7} + \sqrt{5}}$.

(1) $x + y$ (2) xy (3) $x^2y + xy^2$ (4) $\dfrac{y}{x} + \dfrac{x}{y}$

6. (1) Simplify $(1 + \sqrt{2} + \sqrt{3})(1 + \sqrt{2} - \sqrt{3})$.

(2) Rationalize the denominator of $\dfrac{1}{1 + \sqrt{2} + \sqrt{3}}$ using the result of (1).

7. ΔABC at the right is a right triangle where $AB = 1$ and $AD = CD = 2$. Find the length of AC.



2 NUMBERS AND SETS

1 Sets

A collection of all objects which satisfy a certain condition is called a **set**. For example, the collection of all divisors of 6 is a set, and the collection of all positive real numbers is also a set.

Since the divisors of 6 are 1, 2, 3, and 6, the set of all divisors* of 6 is denoted by

$$\{1, 2, 3, 6\}.$$

Generally, a set consisting of objects which can be designated as a_1, a_2, ... is denoted by

$$\{a_1, a_2, \dots\}$$

Example 1 The set of all positive even numbers less than or equal to 10 is $\{2, 4, 6, 8, 10\}$.

Example 2 The set of all natural numbers is $\{1, 2, 3, \dots\}$.

When the contents of a set can be inferred clearly, as in Example 2, we can write out a part of the set inside the curly brackets $\{\ \}$, and represent the remaining elements as ...

Problem 1 Represent the following sets as described above.

(1) All positive odd numbers less than or equal to 10.

(2) All prime numbers less than or equal to 10.

(3) All solutions to the equation $(x-1)(x+2)=0$.

---

* Here, "divisors" means the positive divisors. If we include the negative divisors, the set of all divisors of 6 is $\{1, 2, 3, 6, -1, -2, -3, -6\}$.

The set of all positive real numbers cannot be represented in this way; however, since it consists of all real numbers x which satisfy the condition $x > 0$, the set of all real numbers is denoted by

$$\{x \mid x > 0\}.$$

Generally, the set of all x which satisfy a certain condition is denoted by

$$\{x \mid \text{the condition which } x \text{ satisfies}\}.$$

Example 3 $\{x \mid x < 0\}$ is the set of all negative real numbers.

Example 4 $\{x \mid x^2 = 4\} = \{2, -2\}$

$\{y \mid y \text{ is a positive odd number}\} = \{1, 3, 5, 7, \ldots\}$

$\{2n \mid 0 \leq n \leq 3, n \text{ is an integer}\} = \{0, 2, 4, 6\}$

Each object included in a set is said to be an **element** of that set. For example, the elements of the set $\{x \mid x^2 = 4\}$ are 2 and -2.

When a is an element of set A, a is said to **belong** to A, and this fact is denoted by

$$a \in A.$$

$a \in A$ is sometimes written as $A \ni a$. If a is not an element of A, we write

$$a \notin A \text{ or } A \not\ni a.$$

Example 5 If we let M be the set of all positive even numbers,

$$2 \in M,\ M \ni 4,\ 3 \notin M,\ M \not\ni 5.$$

Problem 2 Fill in the blanks with $\in$ or $\notin$.

(1) $5 \square A, 6 \square A$ for $A = \{x \mid x \text{ is a positive odd number}\}$.

(2) $5 \square B, 6 \square B$ for $B = \{x \mid x \text{ is a divisor of } 12\}$.

2 The Intersection and Union of Sets

The Intersection of Sets

The numbers belonging to both of the sets

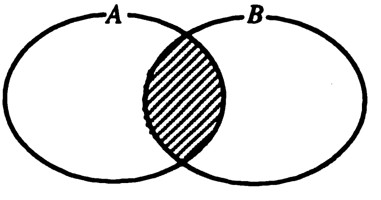
$$A = \{x \mid x \text{ is a divisor of } 8\} = \{1, 2, 4, 8\}$$

$$B = \{x \mid x \text{ is a divisor of } 12\} = \{1, 2, 3, 4, 6, 12\}$$

make up the set

$$\{1, 2, 4\}.$$

The elements of this set are the common divisors of 8 and 12.



In general, the set of all objects belonging to both sets A and B is called the **intersection** of A and B, and is denoted by $A \cap B$.

$$A \cap B = \{x \mid x \in A, x \in B\}$$

Example 1 If $A = \{x \mid x \text{ is a divisor of } 8\} = \{1, 2, 4, 8\}$

$B = \{x \mid x \text{ is a divisor of } 12\} = \{1, 2, 3, 4, 6, 12\}$

then

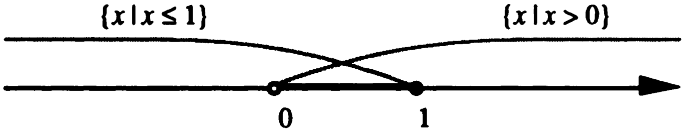
$A \cap B = \{x \mid x \text{ is a common divisor of 8 and } 12\} = \{1, 2, 4\}.$

Example 2 $\{x \mid x \text{ is a multiple of } 4\} \cap \{x \mid x \text{ is a multiple of } 6\}$

$= \{x \mid x \text{ is a common multiple of 4 and } 6\}$

$= \{x \mid x \text{ is a multiple of } 12\}$

Example 3 $\{x \mid x > 0\} \cap \{x \mid x \le 1\} = \{x \mid 0 < x \le 1\}$



Problem 1 Find the following sets:

(1) $\{1, 2, 3, 4\} \cap \{2, 4, 6\}$ (2) $\{1, 2, 4, 8\} \cap (1, 3, 9\}$

(3) $\{1, 2, 3, 6\} \cap \{2, 3\}$ (4) $\{8, 4, 2\} \cap \{1, 2, 4\}$

Problem 2 Given $A = \{x \mid x \le 2\}$, $B = \{x \mid -1 < x\}$, and $C = \{x \mid -2 \le x < 1\}$, which of sets 1 – 6 below represent $A \cap B$, $A \cap C$, and $B \cap C$?

(1) $\{x \mid -2 \le x < -1\}$ (2) $\{x \mid -2 \le x < 1\}$

(3) $\{x \mid -1 < x \le 2\}$ (4) $\{x \mid -1 < x < 1\}$

(5) $\{x \mid |x| < 1\}$ (6) $\{x \mid |x| \le 1\}$

If we let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$, then the intersection of A and B, $A \cap B$, has no elements. Let us call a set with no elements the **empty set** and denote it by the symbol ϕ.

Example 4

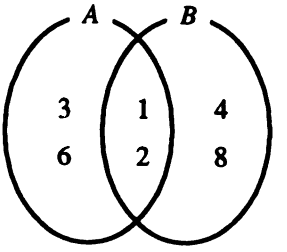
$\{2, 4, 6\} \cap \{1, 3, 5\} = \phi$

$\{x \mid x < 0\} \cap \{x \mid x \ge 1\} = \phi$

The Union of Sets

The numbers which belong to at least one of the two sets $A = \{1, 2, 3, 6\}$ and $B = \{1, 2, 4, 8\}$ make up the set

$$\{1, 2, 3, 4, 6, 8\}.$$



In general, given two sets A and B, the set of objects which belong to at least one of the two sets is called the **union** of A and B, and is denoted by $A \cup B$. So

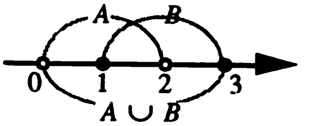
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Here the meaning of "$x \in A$ or $x \in B$" includes the case where x belongs to both A and B.

Example 5 $\{1, 2, 3, 6\} \cup \{1, 2, 4, 8\} = \{1, 2, 3, 4, 6, 8\}$

Example 6 If

$$A = \{x \mid 0 < x < 2\} \text{ and}$$
$$B = \{x \mid 1 \leq x \leq 3\}, \text{ then}$$
$$A \cup B = \{x \mid 0 < x \leq 3\}.$$



Example 7 $\{0\} \cup \{x \mid x > 0\} = \{x \mid x \geq 0\}$

Problem 3 Find the following sets:

(1) $\{3, 2, 1\} \cup \{1, 2, 4, 8\}$

(2) $\{1, 3, 9\} \cup \{1, 3, 5, 7, 9\}$

(3) $\{x \mid 0 < x < 2\} \cup \{x \mid 1 < x\}$

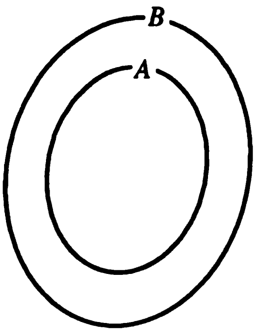
(4) $\{x \mid 2 < x < 4\} \cup \{4\}$

1 Subsets

Given two sets A and B, when each element of A also belongs to B, or

$$x \in A \Rightarrow x \in B,$$

A is said to be a **subset of** B and is denoted by



$$A \subseteq B \text{ or } B \supseteq A.$$

We read this notation "B contains A" or "A is contained in B."

Set A is a subset of A. We can write $A \subseteq A$.

The empty set is a subset of any set A. We can write $\emptyset \subseteq A$.

Example (1) $\{1, 4\} \subseteq \{4, 2, 1\}$

(2) $\{x \mid x \text{ is a multiple of } 6\} \subseteq \{x \mid x \text{ is a multiple of } 3\}$

(3) $\{2, 3\} \subseteq \{x \mid (x-2)(x-3) = 0\}$

Problem 1 Which sets below are subsets of the set $\{1, 2, 3, 6\}$? Which sets contain the set $\{1, 3\}$ as a subset?

(1) $\{1, 2, 3\}$ (2) $\{1, 2, 4\}$

(3) $\{3\}$ (4) $\emptyset$

(5) $\{x \mid x \text{ is a divisor of } 3\}$ (6) $\{x \mid x \text{ is a divisor of } 6\}$

We can represent the fact that $A \subseteq B$ but not $A = B$ by

$$A \subset B \quad \text{or} \quad B \supset A$$

and A is referred to as a **proper subset** of B.

Problem 2 Find all the subsets of the set $\{1, 2\}$. Which is not a proper subset?

Given sets A and B, if $A \subseteq B$ and $B \subseteq A$, then both of the following implications hold:

$$x \in A \Rightarrow x \in B$$

$$x \in B \Rightarrow x \in A.$$

Hence,

$$x \in A \Leftrightarrow x \in B.$$

So the elements of A are identical to the elements of B, and A and B are the same set.

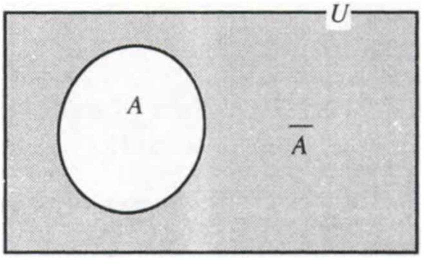
Therefore,

$$A \subseteq B,\ B \subseteq A \Rightarrow A = B.$$

1 Complementary Sets

When we work with sets, we sometimes pick one set U and work only with subsets of U. Set U is referred to as the **universal set**.

For example, when we work with divisors or multiples, we usually take the set of all natural numbers as the universal set. When solving linear equations or inequalities, we usually take the set of all real numbers as the universal set.



For any subset A of the universal set U, the set of all the elements of U which do not belong to A is referred to as the **complement** of A and is denoted by $\overline{A}$.

Example 1 Taking the natural numbers from 1 through 10 as the universal set, the complement of the set

$$\{2, 4, 6, 8, 10\}$$

is

$$\{1, 3, 5, 7, 9\}.$$

Example 2 If we take the set of all real numbers as the universal set and let

$$A = \{x \mid x \text{ is a rational number}\},$$

then

$$\overline{A} = \{x \mid x \text{ is an irrational number}\}.$$

Problem 1 Take $\{1, 2, 3, 4, 5\}$ as the universal set, and find the complements of the following sets:

(1) $\{1, 3, 5\}$ (2) $\{1, 5\}$ (3) $\{1, 2, 3, 4, 5\}$ (4) $\emptyset$

Problem 2 Take the set of all real numbers as the universal set, and find the complement $\overline{A}$ of set $A = \{x \mid x > 1\}$.

If we represent any element of a universal set by x, the complement $\overline{A}$ of set A is denoted by

$$\overline{A} = \{x \mid x \notin A\}.$$

Moreover, the complement of the set $\overline{A}$ is A, so

$$A \cap \overline{A} = \emptyset, \quad A \cup \overline{A} = U.$$

Problem 3 Take the set of all natural numbers from 1 through 10 as the universal set, and let

$$A = \{x \mid x \text{ is a multiple of } 2\} \text{ and } B = \{x \mid x \text{ is a multiple of } 3\}.$$

(1) Find the sets $\overline{A}$, $\overline{B}$, $A \cap B$, $A \cup B$, $\overline{A} \cap \overline{B}$, and $\overline{A} \cup \overline{B}$.

(2) Check that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

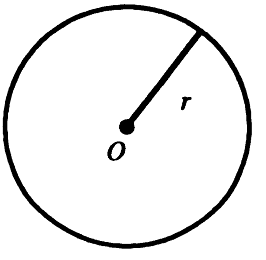
(3) Check that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

❶ Sets with Non-Numerical Elements

So far we have considered only sets whose elements are numbers; however, other sets also occur in mathematics.

Example 1 A circle with its center at a and a radius of r is the set of all points in the plane at a distance of r from point O. If we take the set of all points in the plane as the universal set, this circle is designated by

$$\{P \mid OP = r\}.$$



The interior of this circle is designated by

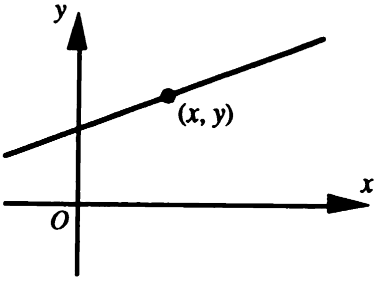
$$\{P \mid OP < r\}$$

and the exterior by

$$\{P \mid OP > r\}.$$

Example 2 The graph of the function $y = ax + b$ is the set of all points (x, y) whose coordinates x and y are real numbers satisfying the equation.

$$\{(x, y) \mid y = ax + b\}$$



Problem Use set symbols to designate the graph of the function $y = x^2$.

Exercises

1. Take $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as the universal set, and let $A = \{2, 3, 5\}$ and $B = \{1, 5, 7, 9\}$. Find the following sets:

   (1) $A \cap B$ (2) $A \cup B$ (3) $\overline{A}$ (4) $A \cap \overline{B}$

2. Given $A = \{x \mid 1 \le x \le 5\}$, $B = \{x \mid 0 < x < 3\}$, and $C = \{x \mid |x| > 2\}$, find the following sets:

   (1) $A \cup B$ (2) $B \cap C$ (3) $\overline{B} \cap C$

3. How many integers from 1 through 100 are multiples of either 3 or 5?

4. Given

   $A = \{x \mid x = 3n + 2, n$ is a natural number less than or equal to $19\}$
   $B = \{x \mid x = 6m - 1, m$ is a natural number less than or equal to $10\}$.

   Represent sets A and B by listing all the elements, and determine which of the following relations holds.

$$A \subseteq B, \ A \supseteq B$$

3 INTEGRAL EXPRESSIONS

2 Integral Expressions

An expression created by multiplying together one or more letters and numbers is called a **monomial**. For example, the following expressions are monomials.

$$5,\ a,\ \frac{1}{3}x^2,\ -2ax^2y,\ bxyz^2$$

The number of letters in a monomial is called the **degree** of the monomial, and the part excluding the letters is called the **coefficient**.

The degree is the sum of the exponents of each letter.

Example 1 The monomial $-2ax^2y$ has one a, two x's, and one y as multiples, so the degree of this monomial is 4 and its coefficient is -2.

The degree of the monomial 5 is 0, and its coefficient is 5. The degree of $bxyz^2$ is 5, and its coefficient is 1.

When a monomial contains more than two different letters, such as $bxyz^2$, we sometimes single out certain letters and say, for example, that the degree of the part involving y and z is 3. Then bx, the remainder of the expression, can be regarded as the coefficient of the monomial.

Example 2 With respect to the letters x and y, the degree of $-2ax^2y$ is 3 and its coefficient is $-2a$. On the other hand, if we single out the letter x, the degree of the monomial is 2 and its coefficient is $-2ay$.

Problem 1 Give the degree and coefficient of the monomial $bxyz^2$ with respect to z. Then give the degree and coefficient with respect to x and y.

An expression which takes the form of the sum of two or more monomials is called a **polynomial**. The monomials that make it up are called the **terms** of the polynomial.

For example, the following expressions are polynomials:

$$3x + x^2, \quad 4xy + (-2z), \quad 3x^2 + axy + (-by^2).$$

Polynomials such as

$$4xy + (-2z), \quad 3x^2 + axy + (-by^2)$$

are usually written in the following form:

$$4xy - 2z, \quad 3x^2 + axy - by^2.$$

Monomials and polynomials together are called **integral expressions.**

Note: The term "polynomials" is sometimes used to mean integral expressions.

Terms that contain the same set of letters, such as

$$2x^2y, \quad \frac{5}{3}x^2y, \quad -6yx^2$$

are called **like terms**. The like terms in one integral expression can be combined into one term as illustrated below.

$$\begin{aligned} 2x^2y + \frac{5}{3}x^2y - 6yx^2 &= (2 + \frac{5}{3} - 6)x^2y \\ &= -\frac{7}{3}x^2y \end{aligned}$$

Combining the like terms is referred to as **simplifying** the integral expression.

Once a given integral expression has been simplified, the greatest degree of any term is referred to as the **degree** of the integral expression. An integral expression whose degree is n is called an ***n*th-degree expression.**

Example 3 $2x + 1$, $\frac{1}{3}x^2 - 1$, $5 + 2x^2 - xy + 4x^2y$ are a linear expression, a quadratic expression, and a cubic expression, respectively.

If we simplify two integral expressions A and B, and the terms of A and the terms of B coincide except for the order in which they are arranged, A and B are said to be **equal**.

For example, $3x - 2 + x^2$ and $x^2 + 3x - 2$ are equal.

Again, we sometimes single out certain letters in an integral expression.

For example, the polynomials

$$ax + b \tag{1}$$

$$ax^2 + bx - 2cx - d^2 + 3 \tag{2}$$

are linear and quadratic expressions, respectively, with respect to x.

By singling out the letter x, expression (2) can be simplified in the following way:

$$ax^2 + bx - 2cx - d^2 + 3 = ax^2 + (b - 2c)x + (-d^2 + 3).$$

A term of an integral expression which does not contain the letters that we have singled out is called a **constant term** or constant.

For example, with respect to x the constant terms in expressions (1) and (2) are b and $-d^2 + 3$, respectively.

Problem 2 What is the degree of each expression? If an expression contains two or more different letters, give the degree with respect to each letter.

(1) $2 + x^2 - x^4$ (2) $a^3 + 3a^2b + 3ab^2 + b^3$

(3) $2x^3y - x^2 + y - 2x^3y$ (4) $\frac{1}{3}x^4y^2z$

Simplifying an expression with respect to one letter and then writing its terms in order from the highest degree term to the lowest degree term is referred to as arranging the expression in **descending order of degree**. Similarly, writing the terms in the opposite order is referred to as arranging the expression in **ascending order of degree**.

Example 4 If we arrange $x - 8 - 7x + 5x^2$ in descending order of degree,

$$\begin{aligned} x - 8 - 7x + 5x^2 &= -6x - 8 + 5x^2 \\ &= 5x^2 - 6x - 8. \end{aligned}$$

If we arrange it in ascending order of degree, it takes the form $-8 - 6x + 5x^2$.

Problem 3 Arrange $5x^2 - 3y + 7xy - x - 4y^2 + 2$ in descending order of degree with respect to x. Then arrange it in ascending order of degree with respect to y.

2 Addition, Subtraction, and Multiplication of Integral Expressions

Addition and Subtraction of Integral Expressions

Demonstration 1 Find $A + B$ and $A - B$ for $A = 4x^3 - 2x^2 + 4$ and $B = 2x^3 + 4x^2 - 3x - 8$.

[Solution]

$$\begin{aligned} A + B &= (4x^3 - 2x^2 + 4) + (2x^3 + 4x^2 - 3x - 8) \\ &= (4 + 2)x^3 + (-2 + 4)x^2 - 3x + (4 - 8) \\ &= 6x^3 + 2x^2 - 3x - 4 \end{aligned}$$

$$\begin{aligned} A - B &= (4x^3 - 2x^2 + 4) - (2x^3 + 4x^2 - 3x - 8) \\ &= 4x^3 - 2x^2 + 4 - 2x^3 - 4x^2 + 3x + 8 \\ &= (4 - 2)x^3 + (-2 - 4)x^2 + 3x + (4 + 8) \\ &= 2x^3 - 6x^2 + 3x + 12 \end{aligned}$$

These problems can also be solved in the following way.

$$\begin{array}{rrrrr} & 4x^3 & -2x^2 & & +4 \\ +) & 2x^3 & +4x^2 & -3x & -8 \\ \hline & 6x^3 & +2x^2 & -3x & -4 \end{array} \qquad \begin{array}{rrrrr} & 4x^3 & -2x^2 & & +4 \\ -) & 2x^3 & +4x^2 & -3x & -8 \\ \hline & 2x^3 & -6x^2 & +3x & +12 \end{array}$$

Problem 1 Find the sum of each pair of integral expressions. Subtract the second expression in each pair from the first. Write the answers in descending order of degree.

(1) $x^3 - 1 - 3x - x^2$, $\quad 4x^3 - 5x - 6x^2 + 1$

(2) $7y - 2y^3 + 6 - 5y^2$, $\quad 8y^2 + 9y - 6y^3 - 3$

Multiplication of Integral Expressions

When we multiply monomials, the following rules apply.

Given that m and n are positive integers,

$$a^m a^n = a^{m+n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n.$$

Example $(-3a^2b^2) \times (2a^2b)^2 = (-3a^2b^2) \times 2^2(a^2)^2b^2$

$= (-3a^2b^2) \times 4a^4b^2$

$= (-3) \bullet 4 \bullet a^2 \bullet a^4 \bullet b^2 \bullet b^2$

$= -12a^6b^4$

Note: $A \bullet B$ has the same meaning as $A \times B$.

Problem 2 Perform the following calculations:

(1) $x \times x^3 \times x^5$ (2) $(-a^2b)^3(2ab)$

(3) $(-x)^2(5x^3y)(-2x^2y)^3$

We generally use the distributive law to multiply integral expressions, as illustrated in Demonstration 2.

Demonstration 2 Calculate $(x^3 - 6x + 1)(2x - 3)$.

[Solution] $(x^3 - 6x + 1)(2x - 3)$

$= (x^3 - 6x + 1)(2x) + (x^3 - 6x + 1)(-3)$

$= 2x^4 - 12x^2 + 2x - 3x^3 + 18x - 3$

$= 2x^4 - 3x^3 - 12x^2 + 20x - 3$

This multiplication can be performed in the following form. We first arrange each expression in descending order of degree or in ascending order of degree.

$$
\begin{array}{rrrrrrl}
 & x^3 & & -\,6x & +\,1 & & \\
\times) & 2x & -\,3 & & & & \\
\hline
 & 2x^4 & & -\,12x^2 & +\,2x & & \leftarrow (x^3-6x+1)(2x) \\
 & & -\,3x^3 & & +\,18x & -\,3 & \leftarrow (x^3-6x+1)(-3) \\
\hline
 & 2x^4 & -\,3x^3 & -\,12x^2 & +\,20x & -\,3 &
\end{array}
$$

In the preceding demonstration the product of the 3rd-degree expression and the expression of degree 1 is the expression of degree 4. In general, the product of an expression of degree m and an expression of degree n is an expression of degree $(m + n)$.

Problem 3 Perform the following multiplications:

(1) $(2x^2 - 4x + 1)(3x - 4)$ (2) $(a^2 - 3a + 5)(a^2 + 4a - 3)$

(3) $(-3x + x^2 - 1)(2 + x - 5x^2)$ (4) $(4x + 3)(2x - x^3 - 1)$

2 Expansion Formulas

Converting expressions in the form of a product of polynomials into the form of a sum of monomials is called **expansion**. The following formulas can be used to perform expansions.

[I]	$(a + b)^2 = a^2 + 2ab + b^2$	Square of a sum.
[II]	$(a - b)^2 = a^2 - 2ab + b^2$	Square of a difference.
[III]	$(a + b)(a - b) = a^2 - b^2$	Product of a sum and a difference.
[IV]	$(x + a)(x + b) = x^2 + (a + b)x + ab$	

[V] $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$

Problem 1 Check formula [V].

Problem 2 Expand the following expressions using the expansion formulas:

(1) $(2a+5)^2$ (2) $(p-2q)^2$

(3) $(x+\frac{y}{2})(x-\frac{y}{2})$ (4) $(3a+4)(2a-5)$

(5) $(x+3y)(x-4y)$

[VI]	$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	Cube of a sum.
[VII]	$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$	Cube of a difference.

[VIII] $(a+b)(a^2-ab+b^2) = a^3 + b^3$

[IX] $(a-b)(a^2+ab+b^2) = a^3 - b^3$

Demonstration 1 Check formula [VI].

[Solution]

$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= (a^2+2ab+b^2)a + (a^2+2ab+b^2)b\\ &= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

Problem 3 Check formulas [VII], [VIII], and [IX].

Problem 4 Expand the following expressions using the expansion formulas:

(1) $(x+3y)^3$ (2) $(2a-3b)^3$

(3) $(x+1)(x^2-x+1)$ (4) $(2x-1)(4x^2+2x+1)$

Formulas [I] and [II] and formulas [VIII] and [IX] can be expressed in a more general form:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a \pm b)(a^2 \mp ab + b^2) = a^3 \pm b^3.$$

The signs $\pm$ and $\mp$ are called **double signs.**

Demonstration 2 Expand $(a + b + c)^2$.

[Solution]

$$\begin{aligned} (a+b+c)^2 &= \{(a+b)+c\}^2 \\ &= (a+b)^2 + 2(a+b)c + c^2 \\ &= a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \\ &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \end{aligned}$$

Problem 5 Expand the following expressions:

(1) $(a + 3b + c)^2$ (2) $(x - y + 1)^2$ (3) $(2a - 3b - c)^2$

4 Factoring

Suppose we have two integral expressions, A and B. If the product of B and a certain other integral expression is A, B is said to be a **factor** of A.

Converting a given integral expression into the product of some factors of degree one or more is called **factoring**. In other words, the operation of factoring is the inverse of expansion.

Extracting the Common Factor

Example (1) $ab + ac - ad = a(b + c - d)$

(2) $$\begin{aligned} x(2y-3) - 2(3-2y) &= x(2y-3) + 2(2y-3) \\ &= (x+2)(2y-3) \end{aligned}$$

Note: Merely extracting a number from each term, as in $2x + 4 = 2(x + 2)$ is not factoring; we usually try to put expressions into the form on the right side so they will be easier to interpret.

Problem 1 Factor the following expressions:

(1) $6a^2b - 3a^3b$ (2) $p^2q - pqr + pq^2$

(3) $5x(x - 3) - 4(3 - x)$ (4) $6x - 3xy - 2 + y$

Factoring Expressions of Degree Two

You have already studied factoring using formulas I – III below in junior high school.

[I] $a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

[II] $a^2 - b^2 = (a + b)(a - b)$

[III] $x^2 + (a + b)x + ab = (x + a)(x + b)$

Problem 2 Factor the following expressions:

(1) $x^2 + 10x + 25$ (2) $a^2b^2 - 4ab + 4$

(3) $9x^2 - 1$ (4) $100 - 4a^2$

(5) $x^2 + 11x + 10$ (6) $x^2 - 12xy + 32y^2$

(7) $c^2 - (a - b)^2$ (8) $32x^4 - 18x^2$

Problem 3 Fill in the blanks with positive numbers to put each expression into the form of a square of an expression of degree one.

(1) $x^2 + 4x + \boxed{}$ (2) $x^2 - 6x + \boxed{}$

(3) $a^2 + \boxed{}\,a + 1$ (4) $4a^2 - \boxed{}\,a + 9$

[IV] $acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$

Demonstration 1 Factor the following expressions using formula [IV]:

(1) $6x^2 - 13x - 15$ (2) $12x^2 - 25xy + 12y^2$

[Solution]

(1) Given $a, b, c,$ and d such that they satisfy the following system of equations:

$$\left.\begin{aligned} ac &= 6 \\ bd &= -15 \end{aligned}\right\}. \quad [1]$$

$$\begin{array}{cccc} \underline{ac} & \underline{bd} & & \underline{ad + bc} \\ a & b & \rightarrow & bc \\ c & d & \rightarrow & \underline{ad} \\ & & & ad + bc \end{array}$$

Find the values which also satisfy the following equation:

$ad + bc = -13.$ [2]

$$\begin{array}{cccc} \underline{6} & \underline{-15} & & \underline{-13} \\ 6 & 5 & \rightarrow & 5 \\ 1 & -3 & \rightarrow & \underline{-18} \\ & & & -13 \end{array}$$

The calculation on the right shows us that [1] and [2] are both satisfied by

$a = 6,\ b = 5,\ c = 1,\ d = -3.$

Therefore,

$6x^2 - 13x - 15 = (6x + 5)(x - 3).$

(2) Given that $12x^2 - 25xy + 12y^2$ is an expression in x of degree 2, the calculation on the right shows us that

$$\begin{array}{cccc} \underline{12} & \underline{12y^2} & & \underline{-25y} \\ 3 & -4y & \rightarrow & -16y \\ 4 & -3y & \rightarrow & \underline{-9y} \\ & & & -25y \end{array}$$

$$\begin{aligned} & 12x^2 - 25xy + 12y^2 \\ = {} & (3x - 4y)(4x - 3y). \end{aligned}$$

Problem 4 Factor the following expressions:

(1) $2x^2 + 3x + 1$ (2) $2x^2 - 13x + 6$

(3) $-3a^2 + 5a + 2$ (4) $6p^2 - 4pq - 16q^2$

(5) $8x^2 - 26xy + 15y^2$ (6) $-35x^2 - 16xy + 12y^2$

Factoring Expressions of Degree Three

Sum and Difference of Cubes

[V] $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Demonstration 2 Factor the following expressions:

(1) $x^3 + 27$ (2) $8a^3 - 1$

[Solution] (1) $x^3 + 27 = x^3 + 3^3$

$= (x + 3)(x^2 - 3x + 9)$

(2) $8a^3 - 1 = (2a)^3 - 1^3$

$= (2a - 1)(4a^2 + 2a + 1)$

Problem 5 Factor the following expressions:

(1) $a^3 - 27$ (2) $64x^3 + 125$

(3) $8p^3 - 125q^3$

Other Kinds of Factoring

So far we have considered only expressions which can be factored directly by applying the formulas. Now let's think about factoring more complicated expressions.

Demonstration 3 Factor the following expression:
$2x^2 - 5xy - 3y^2 + 3x + 5y - 2.$

[Solution] Arranging it in descending order of degree in x, we obtain

$$2x^2 - (5y - 3)x - (3y^2 - 5y + 2)$$
$$= 2x^2 - (5y - 3)x - (y - 1)(3y - 2).$$

$$\begin{array}{cccc}
\underline{2} & \underline{-(y-1)(3y-2)} & & \underline{-(5y-3)} \\
2 & y-1 & \rightarrow & y-1 \\
1 & -(3y-2) & \rightarrow & \underline{-6y+4} \\
 & & & -(5y-3)
\end{array}$$

Then, by reasoning as in the calculation above,

$$2x^2 - (5y - 3)x - (y - 1)(3y - 2)$$
$$= \{2x + (y - 1)\}\{x - (3y - 2)\}$$
$$= (2x + y - 1)(x - 3y + 2).$$

Problem 6 Factor the following expressions:

(1) $x^2 + 3xy + 2y^2 + x - y - 6$

(2) $2x^2 - 3xy - 2y^2 + x + 3y - 1$

Demonstration 4 Factor the following expressions:

(1) $x^4 - 13x^2 + 36$ (2) $(x^2 + x + 2)(x^2 + x - 3) - 6$

[Solution] (1) Let $x^2 = X$, and then

$$\begin{aligned} x^4 - 13x^2 + 36 &= X^2 - 13X + 36 \\ &= (X - 9)(X - 4) \\ &= (x^2 - 9)(x^2 - 4) \\ &= (x + 3)(x - 3)(x + 2)(x - 2). \end{aligned}$$

(2) Let $x^2 + x = X$, and then

$$\begin{aligned} &(x^2 + x + 2)(x^2 + x - 3) - 6 \\ &= (X + 2)(X - 3) - 6 \\ &= X^2 - X - 12 \\ &= (X + 3)(X - 4) \\ &= (x^2 + x + 3)(x^2 + x - 4). \end{aligned}$$

Problem 7 Factor the following expressions:

(1) $(x - y)^2 - 4(x - y) + 4$ (2) $(a + 1)^3 + (a - 1)^3$

(3) $x^4 + x^2y^2 - 2y^4$

If we factor $x^4 - 4$, we obtain

$$x^4 - 4 = (x^2 + 2)(x^2 - 2).$$

Neither $x^2 + 2$ nor $x^2 - 2$ can be factored any further with rational coefficients. An integral expression which cannot be factored any further is called an **irreducible** integral expression.

Usually we factor an integral expression until we get a product of integral expressions which are irreducible in the realm of rational coefficients. If we factor $x^4 - 4$ with real coefficients, we get the following result:

$$x^4 - 4 = (x^2 + 2)(x + \sqrt{2})(x - \sqrt{2}).$$

Problem 8 Factor the following expressions in the realm of real numbers:

(1) $x^2 - 5$ (2) $2x^2 - 3$

Exercises

1. Let $A = x^2 + xy - 3y^2$, $B = 2x^2 - xy + 4y^2$, and $C = -3x^2 + 5y^2$, and simplify the following expressions:

   (1) $A - B + C$ (2) $A - 2B + 3C$ (3) $3(-A + B) + 2C$

2. Expand the following expressions:

   (1) $(2a - 3b)(4a + 5b)$ (2) $(3x - 1)(x^2 + 7x - 5)$

   (3) $(x + 1)(x - 1)(x^2 + 1)$ (4) $(2a - b + 3c)^2$

   (5) $(x - \frac{1}{2})^3$ (6) $(a - 2)(a^2 + 2a + 4)$

3. Expand the following expressions:

   (1) $(a + b - 3)(a + b + 1)$ (2) $(x + y - z)(x - y + z)$

   (3) $(x + y)^2(x - y)^2$ (4) $(x + 2)(x - 6)(x - 8)(x + 4)$

4. Factor the following expressions:

   (1) $3a^2b - 18ab^2$ (2) $x^2 - 11x + 24$

   (3) $x(x - 4) - 5$ (4) $6x^2 + 13x - 8$

   (5) $(a - 3)^2 - (a - 3)$ (6) $x^2 - x - y^2 - y$

   (7) $1 + a - 2b - 2ab$

5. Factor the following expressions:

   (1) $(x - y)(x - y + 5) + 6$ (2) $a^2 - 2ab + b^2 - 9$

   (3) $x^2 - (4a - 3b)x - 12ab$ (4) $4(x - 3y)^2 - 9(x - 3y) + 5$

   (5) $(2x - y)^3 - (2y - x)^3$ (6) $x^2 + xy - 6y^2 + 5x + 35y - 36$

4 DIVISION OF INTEGRAL AND FRACTIONAL EXPRESSIONS

2 Division of Integral Expressions

Given two integral expressions A and B, if there is an integral expression Q such that

$$A = BQ,$$

A is said to be **divisible** by B, B is a **divisor** or factor of A, and A is a **multiple** of B. Moreover, Q is referred to as the **quotient** of A divided by B.

Example Since

$$6x^2 + 5x - 4 = (2x - 1)(3x + 4),$$

we know that

$$(6x^2 + 5x - 4) \div (2x - 1) = 3x + 4.$$

The expression $2x - 1$ is a divisor of $6x^2 + 5x - 4$, and $6x^2 + 5x - 4$ is a multiple of $2x - 1$.

Note: When working with integral expressions, we use such terms as divisor and multiple, as we do with numbers, but we must specify that they are expressions.

The expression $6x^2 + 5x - 4$ in the example can also be given as $(6x - 3)(x + \frac{4}{3})$. Therefore, $6x - 3$ is also a divisor of $6x^2 + 5x - 4$. In general, if an integral expression B is a divisor of integral expression A, any constant multiple of B is also a divisor of A.

Division of an integral expression by an integral expression as in the preceding example can be carried out in the following way:

$$\begin{array}{r r l}
 & 3x \; + \; 4 & \\
\cline{2-2}
2x - 1 \,\big) & 6x^2 + 5x - 4 & \\
 & 6x^2 - 3x & \leftarrow (2x - 1) \cdot 3x \\
\cline{2-2}
 & 8x - 4 & \\
 & 8x - 4 & \leftarrow (2x - 1) \cdot 4 \\
\cline{2-2}
 & 0 &
\end{array}$$

When performing such calculations, it is important to arrange both the divisor expression and the dividend expression in descending order of degree in the specific variable.

Problem 1 Do the following division problems:

(1) $(x^3 + x^2 - 3x - 6) \div (x - 2)$

(2) $(6y^4 - 10y^2 + y^3 + 6 + 7y) \div (2y + 3)$

(3) $(5a^2 + 2a^3 - 1) \div (-1 + 2a + a^2)$

Next, let us divide $2x^3 - 3x^2 - x + 4$ by $2x^2 + 2x - 3$ in the same way.

$$
\begin{array}{r|l}
 & \quad x - \frac{5}{2} \\
\cline{2-2}
2x^2 + 2x - 3 & \; 2x^3 - 3x^2 - x + 4 \\
 & \; \underline{2x^3 + 2x^2 - 3x } \\
 & \qquad -5x^2 + 2x + 4 \\
 & \qquad \underline{-5x^2 - 5x + \frac{15}{2}} \\
 & \qquad\qquad 7x - \frac{7}{2}
\end{array}
$$

We obtain

$$2x^3 - 3x^2 - x + 4 = (2x^2 + 2x - 3)(x - \frac{5}{2}) + (7x - \frac{7}{2}).$$

In general, given two integral expressions A and B of degree m and n, respectively, in the same variables, if $m \geq n$, we can find integral expressions Q and R such that

$$A = BQ + R, \qquad \text{where the degree of } R < \text{the degree of } B,$$

using the kind of calculation above. Q and R are called the **quotient** and the **remainder**, respectively. The degree of the quotient Q is $m - n$.

In particular, when A is divisible by B, the remainder R is 0.

Problem 2 Do the following division problems:

(1) $(3x^2 + 2x + 1) \div (3x - 4)$

(2) $(x^3 - x^2 + x - 1) \div (x + 2)$

(3) $(4x - 5) \div (x + 2)$

(4) $(y^4 - 2y^2 - y + 8) \div (y^2 - y - 2)$

Topic for Enrichment: Synthetic Division

There is a very simple method for dividing an integral expression by a linear expression of the form $x - \alpha$.

For example, to divide $a_0x^3 + a_1x^2 + a_2x + a_3$ by $x - \alpha$, let the quotient be $b_0x^2 + b_1x + b_2$ and the remainder be R.

Since $a_0x^3 + a_1x^2 + a_2x + a_3 = (x - \alpha)(b_0x^2 + b_1x + b_2) + R$,

$$\begin{cases} a_0 = b_0 \\ a_1 = b_1 - \alpha b_0 \\ a_2 = b_2 - \alpha b_1 \\ a_3 = R - \alpha b_2 \end{cases}.$$

Therefore, b_0, b_1, b_2, and R can be obtained separately from the following formulas:

$$b_0 = a_0, \qquad b_1 = a_1 + \alpha b_0, \qquad b_2 = a_2 + \alpha b_1, \qquad \text{and } R = a_3 + \alpha b_2.$$

This calculation is usually performed as illustrated below.

$$\begin{array}{r|cccc}
\alpha & a_0 & a_1 & a_2 & a_3 \\
 & & \alpha b_0 & \alpha b_1 & \alpha b_2 \\
\hline
 & b_0 & b_1 & b_2 & \boxed{R} \\
 & \uparrow & \uparrow & \uparrow & \uparrow \\
 & a_0 & a_1 + \alpha b_0 & a_2 + \alpha b_1 & a_3 + \alpha b_2
\end{array}$$

This method is called **synthetic division.**

For example, if we calculate $(3x^3 + 2x^2 - 6x - 1) \div (x - 2)$ by synthetic division, we get

$$\begin{array}{r|cccc}
2 & 3 & 2 & -6 & -1 \\
 & & 6 & 16 & 20 \\
\hline
 & 3 & 8 & 10 & \boxed{19}
\end{array}$$

Quotient: $3x^2 + 8x + 10$ Remainder: 19

2 The Greatest Common Divisor and Least Common Multiple of Integral Expressions

When we have two or more integral expressions, any divisor common to all of them is called a **common divisor** of these integral expressions, and the common divisor of the highest degree is called the **greatest common divisor**.

A multiple common to two or more integral expressions is called a **common multiple** of these expressions. Aside from zero, the common multiple of the lowest degree is called the **least common multiple**.

For example, given $x^2(x-1)$ and $x(x-1)(x+1)$, it is clear that the greatest common divisor is $x(x-1)$ and the least common multiple is $x^2(x-1)(x+1)$.

$$\begin{array}{c|c|l}
x & x(x-1) & \\
 & x(x-1) & (x+1) \\
\hline
 & x(x-1) & \text{----------- greatest common divisor} \\
x \,\bullet & x(x-1) & \bullet\ (x+1) \text{ ---- least common multiple}
\end{array}$$

Moreover, as you see from the diagram above, the product of the greatest common divisor and the least common multiple of two integral expressions is equal to the product of the two expressions.

In order to find the greatest common divisor and the least common multiple, it is helpful to factor each expression into a product of prime factors.

Demonstration 1 Find the greatest common divisor and the least common multiple of $4x^2 - 2x - 12$ and $4x^2 - 16$.

[Solution] Since

$$4x^2 - 2x - 12 = 2(x-2)(2x+3)$$

$$4x^2 - 16 = 4(x-2)(x+2),$$

the greatest common divisor is $x - 2$

the least common multiple is $(x-2)(x+2)(2x+3)$.

In Demonstration 1, if we let α be a non-zero constant,

$$\alpha(x - 2)$$

is also the greatest common divisor. However, when we find the greatest common divisor, we neglect the possibility of choosing different constants. We can do the same for the least common multiple as well.

If two integral expressions have no common factor of degree one or more, that is, if the greatest common divisor is 1, these two expressions are said to be **relatively prime**.

For example, $(x-1)(x-3)$ and $x+1$ are relatively prime.

Problem 1 Find the greatest common divisor and the least common multiple for each pair of integral expressions:

(1) $x^2 + 7x + 10,\quad x^2 - 25$

(2) $x^2 + 3x + 2,\quad 2x^2 + x - 1$

(3) $x^2 - x - 2,\quad -x^2 - x + 2$

Demonstration 2 Find the greatest common divisor and the least common multiple for the following three integral expressions:

$$x^2 + x - 6, \qquad x^2 + 4x + 3, \qquad x^2 + 3x.$$

[Solution] Factoring each expression, we obtain

$$x^2 + x - 6 = (x+3)(x-2)$$

$$x^2 + 4x + 3 = (x+3)(x+1)$$

$$x^2 + 3x = x(x+3).$$

Therefore,

the greatest common divisor is $x + 3$

the least common multiple is $x(x+3)(x-2)(x+1)$.

Problem 2 Find the greatest common divisor and the least common multiple for each group of integral expressions:

(1) $a^3 - 2a^2$, $a^3 - 4a$, $a^3 - a^2 - 2a$

(2) $(y+z)^2 - x^2$, $(z+x)^2 - y^2$, $(x+y)^2 - z^2$

2 Calculations Involving Fractional Expressions

Suppose A and B are integral expressions. Then an expression of the form $\frac{A}{B}$ is called a **fractional expression** or **rational expression**, and A is the **numerator** and B is the **denominator**.

The fractional expressions $\frac{A}{B}$ and $\frac{A'}{B'}$ are identical when the following equality holds:

$$AB' = A'B.$$

When A is divisible by B, $\frac{A}{B}$ expresses the quotient. The fractional expression $\frac{A}{1}$ is equivalent to the integral expression A.

In general, the following equality is valid:

$$\frac{AC}{BC} = \frac{A}{B}.$$

Therefore, if the numerator and denominator have common divisors of degree one or more, we can **reduce a fraction to lower terms**. A fractional expression that cannot be reduced -- in other words, a fractional expression whose numerator and denominator are relatively prime -- is called an **irreducible fractional expression**. If we reduce a fractional expression by dividing the numerator and denominator by their greatest common divisor, it becomes an irreducible fractional expression.

Demonstration 1 Reduce $\frac{2x^2 - x - 1}{x^3 - 1}$ to its lower terms, and change it into an irreducible fractional expression.

[Solution] $$\frac{2x^2 - x - 1}{x^3 - 1} = \frac{(x-1)(2x+1)}{(x-1)(x^2+x+1)} = \frac{2x+1}{x^2+x+1}$$

Problem 1 Reduce the following expressions and transform them into irreducible fractional expressions:

(1) $\dfrac{3a^2b^3c}{9ab^4c^3}$ (2) $\dfrac{x^2-1}{x^3+1}$ (3) $\dfrac{a^3-a^2-2a}{a^3-4a}$

For example, $\dfrac{2}{x+1}$ and $\dfrac{3}{x-1}$ can be reduced to fractional expressions with the same denominator in the following way:

$$\frac{2}{x+1} = \frac{2(x-1)}{(x+1)(x-1)} \text{ and } \frac{3}{x-1} = \frac{3(x+1)}{(x-1)(x+1)}.$$

Equalizing the denominators of fractional expressions is called **reducing the fractions to a common denominator**. When we reduce fractions to a common denominator, it is convenient to take the least common multiple of the denominators as the common denominator.

Problem 2 Reduce each group of fractional expressions to a common denominator:

(1) $\dfrac{x^2+x+1}{x+1}, \dfrac{x^2-x+1}{x-1}$

(2) $\dfrac{x}{yz}, \dfrac{y}{zx}, \dfrac{z}{xy}$

(3) $\dfrac{1}{x+y}, \dfrac{1}{y-x}, \dfrac{1}{x^2-y^2}$

Addition and Subtraction

Addition and subtraction of fractional expressions with the same denominator can be carried out in the following way.

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C} \text{ and } \frac{A}{C} - \frac{B}{C} = \frac{A-B}{C}.$$

When the denominators are different, first reduce them to a common denominator, and then proceed as described above.

Demonstration 2 Calculate $\frac{x+8}{x^2+x-2} - \frac{x+4}{x^2+3x+2}$.

[Solution]

$$\frac{x+8}{x^2+x-2} - \frac{x+4}{x^2+3x+2}$$

$$= \frac{x+8}{(x+2)(x-1)} - \frac{x+4}{(x+2)(x+1)}$$

$$= \frac{(x+8)(x+1)-(x+4)(x-1)}{(x+2)(x-1)(x+1)}$$

$$= \frac{6x+12}{(x+2)(x-1)(x+1)}$$

$$= \frac{6}{(x-1)(x+1)}$$

Note: A result obtained by performing operations involving fractional expressions must be reduced to an irreducible fractional expression.

Problem 3 Perform the following calculations:

(1) $\frac{x}{x^2-1} - \frac{1}{x^2-1}$ (2) $\frac{x+2}{x-2} + \frac{4}{2-x}$

(3) $\frac{1}{1+a} + \frac{2a}{1-a^2}$

(4) $\frac{x-2}{x^2-x+1} - \frac{1}{x+1} + \frac{x^2+x+3}{x^3+1}$

Problem 4 Check that $\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \frac{4}{x^4+1} = \frac{8}{x^8-1}$.

Multiplication and Division

Multiplication and division of fractional expressions are performed in the following way:

$$\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}$$

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C} = \frac{AD}{BC}.$$

Demonstration 3 Perform the following calculation:

$$\frac{x^2 - 2x - 3}{x^2 - 4} \div \frac{x^2 - 4x + 3}{x^2 + 4x + 4}.$$

[Solution]

$$\frac{x^2 - 2x - 3}{x^2 - 4} \div \frac{x^2 - 4x + 3}{x^2 + 4x + 4} = \frac{x^2 - 2x - 3}{x^2 - 4} \times \frac{x^2 + 4x + 4}{x^2 - 4x + 3}$$

$$= \frac{(x-3)(x+1)}{(x+2)(x-2)} \times \frac{(x+2)^2}{(x-3)(x-1)}$$

$$= \frac{(x+1)(x+2)}{(x-1)(x-2)}$$

Problem 5 Perform the following calculations:

(1) $\dfrac{x^2 - 49}{x^2 + 2x} \times \dfrac{x + 2}{x - 7}$ (2) $\dfrac{2a - 1}{a^2 - 2a + 1} \times \dfrac{a^2 - 3a + 2}{4a^2 - 1}$

(3) $\dfrac{x^2 - 4}{x^2 + 4x + 4} \div \dfrac{x^3 - 8}{2x + 4}$

(4) $\dfrac{2x^3 + 7x^2 + 6x}{3x^2 + 5x - 2} \div \dfrac{4x^3 - 9x}{6x^2 - 11x + 3}$

Problem 6 Simplify the following expressions:

(1) $\dfrac{x - \dfrac{1}{x}}{1 - \dfrac{1}{x}}$ (2) $\dfrac{a + 2}{a - \dfrac{2}{a + 1}}$

2 Expansion of Exponents

Let $a \neq 0$. If we let m and n be positive integers, the following relations hold:

For $m > n$, $a^m \div a^n = a^{m-n}$.

For $m = n$, $a^m \div a^n = 1$.

For $m < n$, $a^m \div a^n = \dfrac{1}{a^{n-m}}$.

In order to express all these relations in one expression that is valid when $m - n$ is equal to 0 or a negative integer, let's define the meaning of a^{m-n}:

$$a^m \div a^n = a^{m-n}. \tag{1}$$

For (1) to be valid for $m = n$, our definition should specify that:

For $a \neq 0$, $a^0 = 1$.

Next, for (1) to be valid for $m < n$,

$$a^{m-n} = a^m \div a^n = \frac{1}{a^{n-m}}.$$

We can now give the following definition:

If $a \neq 0$ and p is a positive integer, $a^{-p} = \dfrac{1}{a^p}$.

Problem 1 Write the following expressions in the form of a^n.

(1) $\dfrac{1}{a}$ (2) 1

(3) $\dfrac{1}{a^7}$ (4) $\dfrac{1}{a^{17}}$

Problem 2 Find the following values:

(1) 2^{-3} (2) 0.5^0

(3) $(-7)^{-2}$ (4) 10^{-5}

Based on our definition of powers with exponents of zero or a negative integer, the following laws are also valid for any integers m and n.

$$a^m a^n = a^{m+n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

These laws are called the **laws of exponents**.

Demonstration For $a \neq 0$, $m = 3$ and $n = -2$, check that the following equalities are true:

(1) $a^m a^n = a^{m+n}$ (2) $(a^m)^n = a^{mn}$

[Solution] (1) By the definition of a negative exponent,

$$a^3 \times a^{-2} = a^3 \times \frac{1}{a^2} = a.$$

But, $$a^{3+(-2)} = a^{3-2} = a.$$

Therefore, $$a^3 \times a^{-2} = a^{3+(-2)}.$$

(2) By the definition of a negative exponent,

$$(a^3)^{-2} = \frac{1}{(a^3)^2} = \frac{1}{a^6}.$$

But, $$a^{3\times(-2)} = a^{-6} = \frac{1}{a^6}.$$

Therefore, $$(a^3)^{-2} = a^{3\times(-2)}.$$

Problem 3 Let $a \neq 0$; for each pair of values of m and n, check that $a^m a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$.

(1) $m = -2, n = -3$ (2) $m = -3, n = 0$

Problem 4 Simplify the following expressions, and give each result without 0 or negative integers as exponents.

(1) $a^{-2} \times a^5$ (2) $a^3 \times a^{-3}$

(3) $x^{-6} \times x^{-4}$ (4) $a^5 \div a^8$

(5) $a^{-5} \div a^3$ (6) $x^0 \div x^{-2}$

(7) $(a^{-2})^{-1}$ (8) $(ab^{-1})^{-3}$

The extraordinarily large and small numbers that occur in physics and other disciplines are often expressed using exponents. For example, one light-year, the distance light travels in one year, is approximately 9.5×10^{15} m, and the mass of an electron is approximately 9.1×10^{-28} g.

Problem 5 The mass of a hydrogen atom is approximately 1.7×10^{-24} g. By approximately what factor is this greater than the mass of an electron?

Exercises

1. Do the following division problems:

   (1) $(4x^3 - 3x - 9) \div (2x - 3)$

   (2) $(a^4 + 2a + 5 - 10a^2 - 4a^3) \div (5a + 4 - a^2)$

2. Find the greatest common divisor and least common multiple for each group of integral expressions:

   (1) $12a^2b^3c, \quad 18a^3bc^2, \quad 30a^4c^2$

   (2) $4x^2 - 9, \quad 6x^2 + 7x - 3, \quad 2x^2 - 17x - 30$

3. Perform the following calculations:

   (1) $\dfrac{a-1}{a} - \dfrac{a}{a-1} + \dfrac{1}{a-1}$

   (2) $\dfrac{1}{x} - \dfrac{1}{x+1} - \dfrac{1}{x+2} + \dfrac{1}{x+3}$

   (3) $\dfrac{2x-1}{x-1} - \dfrac{x+2}{x+1} - \dfrac{x^2+3}{x^2-1}$

   (4) $\dfrac{a^2+3a+2}{a^2+2a+4} \div \dfrac{(a+1)^2}{a^3-8} \times \dfrac{a^2+4a+3}{a^2+a-2}$

4. When integral expression P in x was divided by $3x^2 + 1$, the quotient was $x^3 - 2$ and the remainder was $4x - 5$. Find the integral expression.

5. When $6x^4 - 7x^3 - 4x^2 + 5x + 3$ was divided by integral expression P, the quotient was $2x^2 - 3x + 1$ and the remainder was $-2x + 5$. Find the integral expression P.

6. Find the following values:

   (1) $8^{-2} \times 2^3$ (2) 0.01^{-2} (3) $2^{-1} \times 2^0 \div 2^{-3}$

Chapter Exercises

A

1. Perform the following calculations:

(1) $(3\sqrt{2}+2\sqrt{3}+\sqrt{6})(3\sqrt{2}-2\sqrt{3}-\sqrt{6})$

(2) $\dfrac{2\sqrt{5}-3\sqrt{2}}{3\sqrt{5}-4\sqrt{2}}$

(3) $\dfrac{\sqrt{6}}{\sqrt{3}-\sqrt{2}}-\dfrac{4\sqrt{3}}{\sqrt{2}+\sqrt{6}}-\dfrac{3\sqrt{2}}{\sqrt{6}-\sqrt{3}}$

(4) $\sqrt{\dfrac{2}{6-\sqrt{35}}}$

(5) $\sqrt{\dfrac{\sqrt{2}+1}{\sqrt{2}-1}}-\sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}+1}}$

2. Let p and q be rational numbers. If $p>0$, $q>0$, and $p\neq q$, demonstrate that the following expressions are also rational numbers.

$$\frac{\sqrt{p}+\sqrt{q}}{\sqrt{p}-\sqrt{q}}+\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}$$

3. Given that $U=\{1,2,3,4,5,6,7,8,9\}$ as a universal set, find the following sets for $A=\{2,4,6,8\}$ and $B=\{2,3,5,7\}$.

(1) $A\cap B$ (2) $\overline{A}\cap\overline{B}$ (3) $A\cup\overline{B}$

4. I was supposed to subtract $2xy-3yz+4zx$ from a certain integral expression. By mistake, I added these expressions together and obtained $2yz+zx-2xy$. Find the correct answer.

5. Factor the following expressions:

(1) $x^3 - x^2y - 30xy^2$ (2) $p^3 - p^2q - pq^2 + q^3$

(3) $x^2 - 2(a-1)x - 2a + 1$ (4) $x^2 + ax - 3bx - 3ab$

(5) $x^2y + y^2z - y^3 - x^2z$ (6) $x^2 + 3xy - 5x + 2y^2 - 7y + 6$

(7) $(x^2 + 2x)^2 - 2(x^2 + 2x) - 3$ (8) $9x^4 - 34x^2y^2 + 25y^4$

6. Find the value of constant a such that the integral expression $x^3 + ax^2 + x + 2 - a$ is divisible by $x^2 + 2x - 1$.

7. For $a^n = 3$, find the value of $\dfrac{a^{2n} - a^{-n}}{a^{2n} + a^{-n}}$.

B

1. Demonstrate that the equality $\sqrt{3 + \sqrt{6}} + \sqrt{3 - \sqrt{6}} = \sqrt{6 + 2\sqrt{3}}$ is true.

2. Compare the values of the two numbers $\dfrac{7}{3 - \sqrt{2}}$ and $\dfrac{1}{\sqrt{5} - 2}$.

3. Given $x = \dfrac{3 + \sqrt{13}}{2}$, find the value of the following expressions:

(1) $x - \dfrac{1}{x}$ (2) $x^2 + \dfrac{1}{x^2}$ (3) $x^3 - \dfrac{1}{x^3}$

4. Find two rational numbers a and b which satisfy the equality

$$\frac{40 + 17\sqrt{5}}{2 + \sqrt{5}} = a + b\sqrt{5}.$$

5. When we subtract integral expression P from $\dfrac{x^3}{x+1}$, the result takes the form of $\dfrac{a}{x+1}$. Find the value of a and integral expression P. Let us assume that a is a constant.

6. Perform the following calculations:

(1) $(a+b+c)^2+(b+c-a)^2+(c+a-b)^2+(a+b-c)^2$

(2) $\dfrac{x-z}{(y-z)(x-y)} - \dfrac{y-z}{(x-y)(z-x)} + \dfrac{x-y}{(z-x)(y-z)}$

(3) $\dfrac{a^2-bc}{(a+b)(c+a)} + \dfrac{b^2-ca}{(b+c)(a+b)} + \dfrac{c^2-ab}{(c+a)(b+c)}$

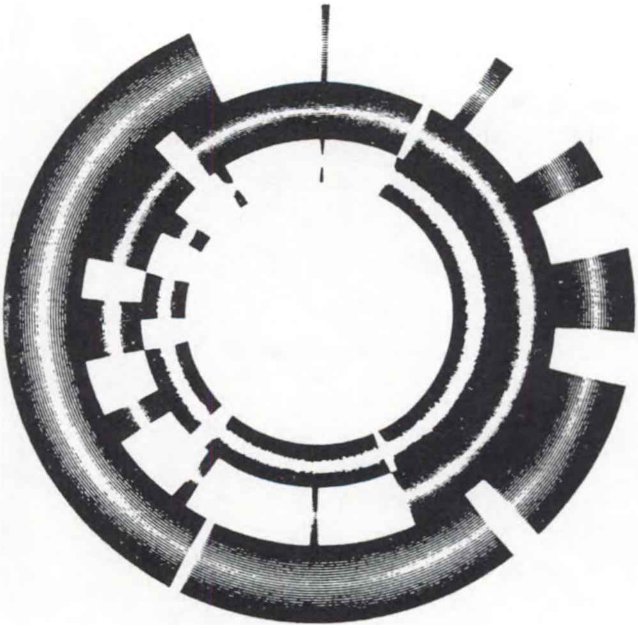
7. (1) Eliminate the double radical in the expression $\sqrt{19-8\sqrt{3}}$, and then simplify it.

(2) Suppose a is an integer and b is a real number greater than or equal to 0 and less than 1. Now, given $\sqrt{19-8\sqrt{3}}=a+b$, find the value of $\dfrac{1}{b}-a$.

8. Find two integral expressions whose greatest common divisor is $x-2$ and whose least common multiple is $3x^3+8x^2-13x-30$.

CHAPTER 2

EQUATIONS AND INEQUALITIES



Solving equations has a very long history. In the fourth and fifth centuries B.C., long before the advent of rigorous mathematics, simple quadratic equations were being solved in ancient Greece and Babylonia by means of tables of numbers. The quadratic formula was stated correctly in words for equations with positive solutions. But the full theory of general quadratic equations was gradually worked out much later.

As you will learn in this chapter, for all quadratic equations to have solutions, we must think in terms of imaginary numbers. For a very long time there was hesitation at treating imaginary numbers as proper numbers. But once the formulas for solving cubic and fourth degree equations were discovered in the early sixteenth century, the efficiency of imaginary numbers gradually came to be appreciated, and in the early nineteenth century, Gauss (1777 – 1855) and other mathematicians established that the concept of complex numbers has direct relevance to real-world phenomena. This paved the way for mathematics to make remarkable advances.

1 QUADRATIC EQUATIONS

1 Quadratic Equations

In junior high school you learned how to solve quadratic equations of the form $ax^2 + bx + c = 0$ by factoring them or by using the **quadratic formula**,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (1)$$

Example Let's solve the quadratic equation $2x^2 - 5x - 3 = 0$ by these two methods.

Factoring the left side,

$$(2x + 1)(x - 3) = 0$$

$$2x + 1 = 0 \text{ or } x - 3 = 0.$$

Therefore, $x = -\frac{1}{2}, 3.$

Or we can use the fact that $a = 2, b = -5, c = -3$, and apply formula (1).

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \bullet 2 \bullet (-3)}}{2 \times 2} = \frac{5 \pm \sqrt{49}}{4} = \frac{5 \pm 7}{4}$$

Therefore, $x = 3, -\frac{1}{2}.$

Problem 1 Solve the following quadratic equations by factoring them:

(1) $x^2 - 5x - 36 = 0$ (2) $4x^2 - 7x + 3 = 0$

Problem 2 Solve the following quadratic equations using the quadratic formula.

(1) $6x^2 - 5x - 4 = 0$ (2) $3x^2 + 7x + 3 = 0$

Since no square roots of negative numbers are real numbers, the right side of the quadratic formula (1) does not express a real number for $b^2 - 4ac < 0$. In other words, for $b^2 - 4ac < 0$, quadratic equations of the form $ax^2 + bx + c = 0$ have no real number solutions. Therefore, for all kinds of quadratic equations to have a solution, we must extend the realm of numbers as described below.

2 Complex Numbers

First, imagine a "new" number which, when squared, yields -1; let us designate this number by the symbol i. Thus,

$$i^2 = -1.$$

Next, imagine a new number expressed in the form

$$a + bi.$$

for any real numbers a and b. Such numbers are called **complex numbers**. For example, the following numbers are all complex:

$$2i,\ -3i,\ 1 + 2i,\ 4 - 5i.$$

Let us give the following definition of equivalent complex numbers. In this section, we can assume that the letters a, b, c, and d represent real numbers.

Equivalent Complex Numbers

Two complex numbers $a + bi$ and $c + di$ are equivalent if and only if both of the following relations hold:

$$a = c \text{ and } b = d.$$

In particular, $a + bi = 0$ is true only if $a = b = 0$.

A complex number $a + 0i$ is equal to the real number a.

A complex number which is not simultaneously a real number, that is, a complex number $a + bi$ with $b \neq 0$, is called an **imaginary number**. And i is called the **imaginary unit**.

> A complex number $a + bi$ is
>
> a real number for $b = 0$
>
> and an imaginary number for $b \neq 0$.

Demonstration 1 Find the value of real numbers x and y such that the following equalities are true:

(1) $2x + (y - 3)i = 0$

(2) $(3x - y) + (2x + 1)i = 7 + 5i$

[Solution]

(1) From the definition of equivalent complex numbers,

$$2x = 0, \quad y - 3 = 0.$$

Therefore, $x = 0, \quad y = 3.$

(2) Similarly,

$$3x - y = 7, \quad 2x + 1 = 5.$$

Therefore, $x = 2, \quad y = -1.$

Problem 1 Find real numbers x and y such that the equality $(3x + 2y) + 9i = 6 - 3yi$ is true.

Let us make the following postulate about calculations involving complex numbers.

> **Calculations Involving Complex Numbers**
>
> Addition, subtraction, multiplication, and division of complex numbers are performed in exactly the same way as with real numbers. Whenever i^2 appears in the process of calculation, it should be replaced with -1.

Demonstration 2

Perform the following calculations for two complex numbers $\alpha = 2 + 3i$ and $\beta = 1 - 5i$:

(1) $\alpha + \beta$ (2) $\alpha - \beta$

(3) $\alpha\beta$ (4) $\frac{\alpha}{\beta}$

[Solution]

(1) $\alpha + \beta = (2 + 3i) + (1 - 5i) = (2 + 1) + (3i - 5i)$

$= 3 - 2i$

(2) $\alpha - \beta = (2 + 3i) - (1 - 5i) = (2 - 1) + (3i + 5i)$
$= 1 + 8i$

(3) $\alpha\beta = (2 + 3i)(1 - 5i) = 2 + 3i - 10i - 15i^2$

$= 2 + 3i - 10i + 15 = 17 - 7i$

(4) $\frac{\alpha}{\beta} = \frac{2 + 3i}{1 - 5i} = \frac{(2 + 3i)(1 + 5i)}{(1 - 5i)(1 + 5i)}$

$= \frac{2 + 3i + 10i + 15i^2}{1 - 25i^2} = \frac{-13 + 13i}{26}$

$= -\frac{1}{2} + \frac{1}{2}i$

In general, when we find a quotient of the complex numbers $\frac{a + bi}{c + di}$, we should multiply $c - di$ by both the numerator and the denominator. The product of $c + di$ and $c - di$ is

$$(c + di)(c - di) = c^2 + d^2.$$

and $c^2 + d^2$ is a positive number as long as $c + di \neq 0$.

Addition, subtraction, multiplication, and division of complex numbers are performed generally in the following way.

(1) $(a + bi) + (c + di) = (a + c) + (b + d)i$

(2) $(a + bi) - (c + di) = (a - c) + (b - d)i$

(3) $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

(4) $\dfrac{a + bi}{c + di} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{bc - ad}{c^2 + d^2}i$

Problem 2 Perform the following calculations:

(1) $(4 + 5i) + (3 - 2i)$ (2) $(2 - 4i) - (1 - i)$

(3) $(5 + 3i)(5 - 3i)$ (4) $\dfrac{-3 + 2i}{2 + 3i}$

Problem 3 Perform the following calculations:

(1) i^3 (2) i^4 (3) $\dfrac{1}{i}$

(4) $3i(1 + 2i)^2$ (5) $(1 - i)^4$ (6) $\dfrac{1 + 2i}{3 - i} + \dfrac{1 - 2i}{3 + i}$

In this way, you can perform four arithmetic operations involving complex numbers, just as with rational numbers or real numbers.

Moreover, for two complex numbers α and β the following generalization holds:

$$\alpha\beta = 0 \Leftrightarrow \alpha = 0 \text{ or } \beta = 0.$$

The complex number $a - bi$ is referred to as the **conjugate** complex number of $a + bi$. The conjugate complex number of $a - bi$ is $a + bi$.

Problem 4 Find the conjugate complex numbers of the following complex numbers:

(1) $6 + 4i$ (2) $2 - 7i$ (3) $\sqrt{2}\,i$ (4) -5

Problem 5 Given β as the conjugate complex number of complex number α. Prove that $\alpha+\beta$ and $\alpha\beta$ are both real numbers.

Square Roots of a Negative Number

We can find the square roots of a negative number by extending the realm of numbers to include complex numbers.

The square root of $-a$ is a number x which satisfies the following condition for a positive number a:

$$x^2 = -a. \tag{1}$$

Since

$$(\sqrt{a}\, i)^2 = ai^2 = -a,$$

equation (1) can be rewritten as

$$x^2 = (\sqrt{a}\, i)^2.$$

Therefore,

$$x^2 - (\sqrt{a}\, i)^2 = 0.$$

Thus,

$$(x - \sqrt{a}\, i)(x + \sqrt{a}\, i) = 0$$

$$x - \sqrt{a}\, i = 0 \text{ or } x + \sqrt{a}\, i = 0.$$

So,

$$x = \sqrt{a}\, i \text{ or } x = -\sqrt{a}\, i.$$

The square roots of a negative number $-a$ are $\sqrt{a}\, i$ and $-\sqrt{a}\, i$.

For a negative number $-a$, among its square roots we denote $\sqrt{a}\, i$ by the symbol $\sqrt{-a}$. In other words,

$$\text{for } a > 0,\ \sqrt{-a} = \sqrt{a}\, i.$$

Therefore, we can represent the two square roots of negative number $-a$ in one expression as $\pm\sqrt{-a}$.

Example $\sqrt{-49} = \sqrt{49}\,i = 7i$ $\qquad \sqrt{-1} = \sqrt{1}\,i = 1i = i$

Problem 6 Express the following numbers using i:

(1) $\sqrt{-8}$ (2) $-\sqrt{-50}$ (3) $\sqrt{-\frac{7}{16}}$

Let us summarize the rules for the square roots we have already studied.

> For $a > 0$,
>
> (1) $\sqrt{a}$ designates the positive square root of a.
>
> (2) $\sqrt{-a} = \sqrt{a}\,i$.
>
> For $a = 0$, $\sqrt{0} = 0$.

Example 3 Perform the folowing calculations:

(1) $\sqrt{-4} + \sqrt{-9}$ (2) $\sqrt{-8} \times \sqrt{-6}$ (3) $\dfrac{\sqrt{-63}}{\sqrt{-7}}$

[Solution] (1) $\sqrt{-4} + \sqrt{-9} = \sqrt{4}\,i + \sqrt{9}\,i = 2i + 3i = 5i$

(2) $\sqrt{-8} \times \sqrt{-6} = \sqrt{8}\,i \times \sqrt{6}\,i = 4\sqrt{3}\,i^2 = -4\sqrt{3}$

(3) $\dfrac{\sqrt{-63}}{\sqrt{-7}} = \dfrac{\sqrt{63}\,i}{\sqrt{7}\,i} = \sqrt{9} = 3$

Problem 7 Perform the following calculations:

(1) $\sqrt{-48} - \sqrt{-12}$ (2) $\sqrt{-28} \times \sqrt{-35}$

(3) $\dfrac{\sqrt{-9}}{3}$ (4) $\dfrac{6}{\sqrt{-4}}$

(5) $(3 + \sqrt{-2})(3 - \sqrt{-2})$

Problem 8 Perform the following calculations, and compare the results:

(1) $\sqrt{-2} \times \sqrt{-5}$, $\sqrt{(-2)(-5)}$

(2) $\sqrt{2} \times \sqrt{-5}$, $\sqrt{2 \times (-5)}$

(3) $\dfrac{\sqrt{-2}}{\sqrt{-5}}, \sqrt{\dfrac{-2}{-5}}$ (4) $\dfrac{\sqrt{2}}{\sqrt{-5}}, \sqrt{\dfrac{2}{-5}}$

3 The Quadratic Formula

If we take into account solutions in the realm of complex numbers, the quadratic equation

$$ax^2 + bx + c = 0 \qquad (1)$$

always has solutions whenever a, b, and c are any real numbers, and these solutions can be found by the quadratic formula. Let's demonstrate how the equation can be solved.

Transposing the constant term c in (1) to the right side, and dividing both sides by a, we obtain

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Let us add $\left(\frac{b}{2a}\right)^2$ to both sides to put the left side into the form of a square.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad (2)$$

Since we can deal with complex number square roots of a negative number, (2) gives us

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

regardless of the value of $b^2 - 4ac$. Therefore,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The Quadratic Formula

The solutions of a quadratic equation

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Demonstration 1

Solve the following equations using the quadratic formula:

(1) $3x^2 - 5x + 1 = 0$

(2) $9x^2 - 12x + 4 = 0$

(3) $4x^2 + 3x + 2 = 0$

[Solution]

(1) From the quadratic formula,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 3 \cdot 1}}{2 \times 3} = \frac{5 \pm \sqrt{13}}{6}.$$

(2) From the quadratic formula,

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \cdot 9 \cdot 4}}{2 \times 9} = \frac{12 \pm \sqrt{0}}{18}$$

$$= \frac{12}{18} = \frac{2}{3}.$$

(3) From the quadratic formula,

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 4 \cdot 2}}{2 \times 4} = \frac{-3 \pm \sqrt{-23}}{8}$$

$$= \frac{-3 \pm \sqrt{23}\, i}{8}.$$

Some quadratic equations have only one solution, like (2) in Demonstration 1. In this case, we consider that there are two coincident solutions, and we call this solution a **multiple solution** or **multiple root**.

Some quadratic equations have solutions which are imaginary numbers, like (3) in Demonstration 1. Solutions which are imaginary numbers are called **imaginary solutions** or **imaginary roots**. On the other hand, solutions which are real numbers are called **real solutions** or **real roots**.

If we take into account imaginary solutions and regard a multiple solution as two solutions, then

a quadratic equation always has two solutions.

Problem 1 Solve the following equations using the quadratic formula:

(1) $x^2 + 3x - 1 = 0$ (2) $3x^2 - 7x + 2 = 0$

(3) $5x^2 - 6x + 4 = 0$ (4) $8x^2 + 13x + 6 = 0$

(5) $8x(3 - 2x) = 9$ (6) $\frac{x^2}{3} - x + \frac{3}{2} = 0$

Problem 2 Demonstrate that the solutions of $ax^2 + 2b'x + c = 0$ are given by the formula

$$x = \frac{-b' \pm \sqrt{b'^2 - ac}}{a}.$$

Problem 3 Solve the following equations using the formula from Problem 2.

(1) $x^2 - 6x - 3 = 0$ (2) $9x^2 + 12x + 7 = 0$

Discriminant

Since the solutions of a quadratic equation of the form

$$ax^2 + bx + c = 0 \qquad (1)$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad (2)$$

we can distinguish between these solutions, whether they are two non-equal real solutions, a multiple solution, or two non-equal imaginary solutions, by means of the sign of the expression

$$b^2 - 4ac$$

under the radical in formula (2).

This expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation (1), and is usually designated by D.

> **Classifying the Solutions of Quadratic Equations According to Their Discriminant**
>
> If we designate the discriminant $b^2 - 4ac$ of a quadratic equation
>
> $$ax^2 + bx + c = 0$$
>
> as D, then:
>
> (1) for $D > 0$, this equation has **two non-equal real solutions.**
>
> (2) for $D = 0$, this equation has a **multiple solution.**
>
> (3) for $D < 0$, this equation has **two non-equal imaginary solutions.**
>
> Combining (1) and (2), we can say
>
> for $D \geq 0$, this equation has **real solutions.**

Note: Two imaginary solutions in (3) are conjugate complex numbers of each other.

For a quadratic equation of the form $ax^2 + 2b'x + c = 0$, since

$$D = 4b'^2 - 4ac = 4(b'^2 - ac),$$

we can replace D with

$$\frac{D}{4} = b'^2 - ac.$$

Demonstration 2 Classify the solutions of the following quadratic equations according to their discriminant:

(1) $2x^2 + 5x - 4 = 0$

(2) $16x^2 - 40x + 25 = 0$

(3) $5x^2 - 3x + 4 = 0$

[Solution] (1) Since

$D = 5^2 - 4 \cdot 2 \cdot (-4) = 57 > 0$

it has two non-equal real solutions.

(2) Since

$\frac{D}{4} = (-20)^2 - 16 \cdot 25 = 0$

it has a multiple solution.

(3) Since

$D = (-3)^2 - 4 \cdot 5 \cdot 4 = -71 < 0$

it has two non-equal imaginary solutions.

Problem 4 Classify the solutions of the following quadratic equations according to their discriminant:

(1) $9x^2 - 24x + 16 = 0$ (2) $5x^2 + 6x = 0$

(3) $4x^2 - 13x - 1 = 0$ (4) $8x^2 + 7x + 3 = 0$

Problem 5 Prove that the quadratic equation $ax^2 + bx + c = 0$ has two non-equal real solutions for $ac < 0$.

Necessary Condition and Sufficient Condition

For any two numbers a and b, if $a = 0$, then $ab = 0$. That is

$$a = 0 \Rightarrow ab = 0. \tag{1}$$

When the relation $p \Rightarrow q$ is true,

q is called a **necessary condition** for p to hold true; and

p is called a **sufficient condition** for q to hold true.

Example 1 Since (1) above is true, $ab = 0$ is a necessary condition for $a = 0$ to be true; and $a = 0$ is a sufficient condition for $ab = 0$ to be true.

Given $p \Rightarrow q$, the relation $q \Rightarrow p$ is said to be its **converse**.

Example 2 The converse of (1) above is

$$ab = 0 \Rightarrow a = 0.$$

This relation is not true.

When $p \Rightarrow q$ and its converse $q \Rightarrow p$ are both true, that is,

$$p \Leftrightarrow q$$

is true, q is called a **necessary and sufficient condition** for p to hold true. Moreover, the conditions p and q are said to be **equivalent**.

Example 3 Since

$$ab = 0 \Leftrightarrow a = 0 \text{ or } b = 0,$$

"$a = 0$ or $b = 0$" is a necessary and sufficient condition for $ab = 0$ to hold true.

Problem 6 What kind of condition is q in (1) – (3) for p to hold true: necessary, sufficent, or necessary and sufficient?

(1) p: $x = 2, y = 3$ q: $x + y = 5$

(2) p: $a = 0$ q: $a^2 = 0$

(3) p: $ac = bc$ q: $a = b$

The following implications hold between the solution of a quadratic equation

$$ax^2 + bx + c = 0 \tag{1}$$

and its discriminant D, which you learned about on page 68.

[1] $D > 0 \Rightarrow$ two non-equal real solutions

[2] $D = 0 \Rightarrow$ a multiple solution

[3] $D < 0 \Rightarrow$ two non-equal imaginary solutions

For each of these three implications [1], [2], and [3], the converse is true. For example, the converse of [1] can be proved in the following way:

Let us assume that quadratic equation (1) has two non-equal real solutions. If we assume that $D > 0$ is not true, then either

$$D = 0 \text{ or } D < 0.$$

For $D = 0$, (1) has a multiple solution by virtue of [2] and for $D < 0$, (1) has two non-equal imaginary solutions because of [3]. These results contradict our original assumption. Therefore, it must be true that $D > 0$. Then

A quadratic equation has two non-equal real solutions $\Rightarrow D > 0$.

The converse of [2] and [3] can be proved analogously.

Therefore,

$D > 0 \Leftrightarrow$ two non-equal real solutions.

$D = 0 \Leftrightarrow$ a multiple solution.

$D < 0 \Leftrightarrow$ two non-equal imaginary solutions.

Demonstration 3 Find the value of the constant a such that the quadratic equation $x^2 + (a + 4)x + a^2 + 5 = 0$ has a multiple solution.

[Solution] Let D be the discriminant of this quadratic equation. Since the necessary and sufficient condition for a multiple solution is that $D = 0$,

$$D = (a + 4)^2 - 4(a^2 + 5) = 0.$$

Therefore,

$$3a^2 - 8a + 4 = 0$$

$$(a - 2)(3a - 2) = 0.$$

Hence,

$$a = 2, \frac{2}{3}.$$

Problem 7 If the quadratic equation in Demonstration 3 has a multiple solution, find it.

Problem 8 Find the value of the constant a such that the quadratic equation $2x^2 - 2ax + a^2 - 8 = 0$ has a multiple solution. Then find the solution as well.

4 Relations between Solutions and Coefficients

From now on, we will assume that the coefficients of an equation are real numbers, and we will take into account solutions of equations in the realm of complex numbers.

If α and β are two solutions of a quadratic equation

$$ax^2 + bx + c = 0,$$

then the quadratic formula gives us

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

$$\alpha\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Therefore, the following relations hold between the solutions and coefficients of a quadratic equation.

Relations between the Solutions and Coefficients of a Quadratic Equation

If α and β are two solutions of a quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

Problem 1 Find the sum and the product of the solutions of the following quadratic equations:

(1) $x^2 - x + 1 = 0$ (2) $2x^2 + 3x - 4 = 0$

Demonstration 1 Let α and β be two solutions of the quadratic equation $2x^2 + 8x + 3 = 0$. Find the following values:

(1) $\alpha^2\beta + \alpha\beta^2$ (2) $\alpha^2 + \beta^2$

[Solution] From the relation between solutions and coefficients

$\alpha + \beta = -\frac{8}{2} = -4,\ \alpha\beta = \frac{3}{2}$ we get

(1) $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta) = \frac{3}{2} \times (-4) = -6$

(2) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (-4)^2 - 2 \times \frac{3}{2} = 13$

Problem 2 Let α and β be two solutions of the quadratic equation $2x^2 - 4x + 5 = 0$. Find the following values:

(1) $(\alpha - \beta)^2$ (2) $\alpha^3 + \beta^3$

(3) $\frac{1}{\alpha} + \frac{1}{\beta}$ (4) $\frac{\beta}{\alpha - 2} + \frac{\alpha}{\beta - 2}$

Factoring Quadratic Expressions

If α and β are two solutions of a quadratic equation $ax^2 + bx + c = 0$, then the quadratic expression $ax^2 + bx + c$ can be factored as

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

[Proof] From the relations between solutions and coefficients $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$ we get

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x + \frac{c}{a}) \\ &= a\{x^2 - (\alpha + \beta)x + \alpha\beta\} \\ &= a(x - \alpha)(x - \beta) \end{aligned}$$

Demonstration 2 Factor $56x^2 - 50x - 25$.

[Solution] If we solve $56x^2 - 50x - 25 = 0$, then

$$x = \frac{25 \pm \sqrt{25^2 + 56 \cdot 25}}{56} = \frac{25 \pm \sqrt{25 \cdot 81}}{56} = \frac{25 \pm 45}{56}.$$

Therefore,

$$x = \frac{5}{4}, -\frac{5}{14}.$$

Hence,

$$56x^2 - 50x - 25 = 56(x - \frac{5}{4})(x + \frac{5}{14})$$

$$= (4x - 5)(14x + 5).$$

Problem 3 Factor the following quadratic expressions:

(1) $x^2 - 7x - 98$ (2) $6x^2 + 11x - 35$

Demonstration 3 Factor the following quadratic expressions in the realm of complex numbers:

(1) $x^2 + 2x - 1$ (2) $2x^2 - 3x + 2$

[Solution] (1) If we solve $x^2 + 2x - 1 = 0$, then

$$x = -1 \pm \sqrt{2}.$$

Therefore,

$$x^2 + 2x - 1 = \{x - (-1 + \sqrt{2})\}\{x - (-1 - \sqrt{2})\}$$

$$= (x + 1 - \sqrt{2})(x + 1 + \sqrt{2}).$$

(2) If we solve $2x^2 - 3x + 2 = 0$, then

$$x = \frac{3 \pm \sqrt{9 - 16}}{4} = \frac{3 \pm \sqrt{7}\, i}{4}.$$

Therefore,

$$2x^2 - 3x + 2 = 2\left(x - \frac{3 + \sqrt{7}\, i}{4}\right)\left(x - \frac{3 - \sqrt{7}\, i}{4}\right).$$

Note: Quadratic equations (1) and (2) above cannot be factored in the realm of rational numbers. Equation (2) cannot be factored even in the realm of real numbers.

Problem 4 Factor the following quadratic equations in the realm of complex numbers:

(1) $x^2 - x - 1$ (2) $x^2 + 4$ (3) $3x^2 - 2x + 1$

The quadratic equation whose two solutions are α and β is

$$(x - \alpha)(x - \beta) = 0.$$

That is,

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

Therefore the following generalization holds.

> A quadratic equation whose solutions are two numbers α and β, where $\alpha + \beta = p$ and $\alpha\beta = q$, takes the form
>
> $$x^2 - px + q = 0.$$

Problem 5 Find the quadratic equations whose solutions are given by the following pairs of numbers:

(1) $2 + \sqrt{5}, 2 - \sqrt{5}$ (2) $\frac{-5 + i}{2}, \frac{-5 - i}{2}$

Demonstration 4 Let α and β be the two solutions of quadratic equation $x^2 - 2x + 7 = 0$. Find the quadratic equation whose solutions are $\alpha + 2$ and $\beta + 2$.

[Solution] Since $\alpha + \beta = 2$ and $\alpha\beta = 7$, we get

$$(\alpha + 2) + (\beta + 2) = (\alpha + b) + 4 = 6$$

$$(\alpha + 2)(\beta + 2) = \alpha\beta + 2(\alpha + \beta) + 4 = 15.$$

Therefore, the quadratic equation we are looking for is

$$x^2 - 6x + 15 = 0.$$

Problem 6 Let α and β be the two solutions of the quadratic equation $2x^2 - x - 5 = 0$. Find the quadratic equation whose solutions are $2\alpha - 1$ and $2\beta - 1$.

Problem 7 Let α and β be the two solutions of the quadratic equation $x^2 + ax + b = 0$. Then prove that the quadratic equation whose solutions are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is expressed as $bx^2 + ax + 1 = 0$. Here we will assume that $b \neq 0$.

Exercises

1. In which cases among (1) – (4) below is the equality $\sqrt{a}\sqrt{b} = \sqrt{ab}$ true? In which cases is the equality $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ true?

   (1) $a > 0, b > 0$ (2) $a > 0, b < 0$

   (3) $a < 0, b > 0$ (4) $a < 0, b < 0$

2. Solve the following quadratic equations:

   (1) $\frac{1}{3}x^2 - x + 2 = 0$ (2) $(x + 1)^2 + (x + 2)^2 = (x - 3)^2$

3. Classify the solutions of the following quadratic equations in x according to their discriminant:

   (1) $-2x^2 + 4x - 3 = 0$ (2) $x(5 - 3x) = 2$

   (3) $3x^2 - mx - 1 = 0$ (4) $x^2 - 4ax + 5a^2 = 0$

4. Fill in the blanks with "necessary", "sufficient", or "necessary and sufficient."

   (1) $ab = 12$ is a [ ] condition for $a = 3, b = 4$.

   (2) $q < 0$ is a [ ] condition for the quadratic equation $x^2 + px + q = 0$ to have two non-equal real solutions.

5. Let α and β be the two solutions of the quadratic equation $x^2 + ax + 3 = 0$. Find the value of the constant a such that the following equalities are true:

   (1) $\alpha^2 + \beta^2 = 3$ (2) $\dfrac{\beta}{\alpha} + \dfrac{\alpha}{\beta} = 10$

6. A rectangle has a length 1 cm longer than its width. Find the length and width if its area is 42 cm^2.

7. Find the quadratic equation whose solutions are obtained by first doubling the two solutions of the quadratic equation $2x^2 - 3x + 2 = 0$, and then adding 1 to each number.

8. Find the value of the constant k such that the ratio of the two solutions of the quadratic equation $2x^2 - kx + k + 2 = 0$ is $3 : 2$.

2 SIMULTANEOUS EQUATIONS AND HIGHER DEGREE EQUATIONS

1 Simultaneous Linear Equations in Three Variables

A linear equation involving three unknowns is called a **linear equation in three variables.** A combination of these equations is called **simultaneous linear equations in three variables.**

Demonstration

Solve the following simultaneous linear equations in three variables:

$$\begin{cases} 3x + y + 2z = 14 & (1) \\ 2x + y - z = 5 & (2) \\ x - 4y + 2z = 3 & (3) \end{cases}$$

[Solution]

Eliminate one variable z to make simultaneous linear equations in x and y. That is,

from (1) + (2) x 2 $\quad 7x + 3y = 24,$ (4)

from (1) – (3) $\quad 2x + 5y = 11.$ (5)

Solving (4) and (5) for x and y, we obtain

$$x = 3, \ y = 1.$$

Then substitute these values of x and y into (1),

$9 + 1 + 2z = 14.$ Hence, $z = 2.$

Answer: $x = 3, y = 1, z = 2$

Problem Solve the following simultaneous equations:

(1) $\begin{cases} 3x + 2y + z = 7 \\ x - y - z = 0 \\ 2x - 3y + z = 10 \end{cases}$ (2) $\begin{cases} x = y - 7 \\ z = x + 5 \\ x + y + z = 6 \end{cases}$

(3) $2x + 3y - z = x + 4y + z = 3x - 2y = 6$

4 Simultaneous Linear and Quadratic Equations

Simultaneous equations in two variables are solved by eliminating one variable using the substitution method.

Demonstration 1 Solve the simultaneous equations $\begin{cases} y = x + 2 \\ x^2 + y^2 = 10 \end{cases}$.

[Solution]

$$\begin{cases} y = x + 2 & (1) \\ x^2 + y^2 = 10 & (2) \end{cases}$$

Substitute (1) into (2):

$$x^2 + (x + 2)^2 = 10.$$

Rearranging this equation,

$$x^2 + 2x - 3 = 0.$$

Therefore, $x = 1$ or $x = -3$.

For $x = 1$, from (1): $y = 3$.

For $x = -3$, from (1): $y = -1$.

Answer: $\begin{cases} x = 1 \\ y = 3 \end{cases}$ $\begin{cases} x = -3 \\ y = -1 \end{cases}$

Problem 1 Solve the following pairs of simultaneous equations:

(1) $\begin{cases} y = x - 1 \\ x^2 + y^2 = 25 \end{cases}$ (2) $\begin{cases} x + 3y = 5 \\ x + y^2 = 3 \end{cases}$ (3) $\begin{cases} y = \sqrt{3}\,x \\ x^2 + y^2 = 48 \end{cases}$

Problem 2 A right triangle has a hypotenuse of 13 cm, and the sum of the lengths of the other two sides is 17 cm. Find the lengths of the two sides that form the right angle.

Problem 3 A rectangular plot of land is x m long and y m wide, and its perimeter is 100 m. We assume that $x > y$. After a square portion of this plot with a side of y m was sold, the remaining area was 200 m^2. Find the values of x and y.

Demonstration 2 Solve the simultaneous equations $\begin{cases} x + y = 10 \\ xy = 24 \end{cases}$.

[Solution] From the relations between the solutions and coefficients of a quadratic equation, we can regard x and y as the solutions of the quadratic equation

$$t^2 - 10t + 24 = 0.$$

Solving this equation, we obtain

$$t = 4,\ 6.$$

Therefore, $\begin{cases} x = 4 \\ y = 6 \end{cases}$ $\begin{cases} x = 6 \\ y = 4 \end{cases}$.

Problem 4 Solve the following simultaneous equations:

(1) $\begin{cases} x + y = 11 \\ xy = 30 \end{cases}$ (2) $\begin{cases} x + y = -\dfrac{20}{3} \\ xy = 4 \end{cases}$

4 Simple Higher Degree Equations

An equation which is represented as

$$(\text{integral expression of degree } n) = 0$$

is called an **equation of degree n.**

For example, the equations

$$x^3 - 3x + 2 = 0, \quad 2x^4 - 3x^2 - x + 2 = 0$$

are a cubic equation and an equation of degree 4, respectively.

Solving equations of a higher degree than 3 is generally very difficult, but some equations can be solved by factoring.

Demonstration 1 Solve the equation $x^3 = 1$.

[Solution] Transposing 1 to the left side, we get

$$x^3 - 1 = 0.$$

Factoring the left side, we get

$$(x - 1)(x^2 + x + 1) = 0$$

$$x - 1 = 0 \text{ or } x^2 + x + 1 = 0.$$

Therefore, $$x = 1, \frac{-1 \pm \sqrt{3}\, i}{2}.$$

Note: All three solutions of the equation in Demonstration 1 are called **cube roots** of 1. A cube root of 1 is a number that yields 1 when it is cubed.

Problem 1 If we let ω designate one imaginary cube root of 1, $\frac{-1 + \sqrt{3}\, i}{2}$, then the other imaginary cube root $\frac{-1 - \sqrt{3}\, i}{2}$ is ω^2. Check this.

Problem 2 Solve the following equations:

(1) $x^3 = 8$ (2) $x^3 = -1$ (3) $x^4 = -8x$

Demonstration 2 Solve the following equations:

(1) $x^3 - 3x^2 - 4x + 12 = 0$ (2) $x^4 - 2x^2 - 8 = 0$

[Solution]

(1) Since the left side is equal to $x^2(x - 3) - 4(x - 3)$, we can factor it to obtain

$$(x^2 - 4)(x - 3) = 0$$

$$(x + 2)(x - 2)(x - 3) = 0.$$

Answer: $x = -2, 2, 3$

(2) Let $x^2 = t$.

$$t^2 - 2t - 8 = 0$$

Therefore, $t = 4$ or $t = -2$.

For $t = x^2 = 4$: $x = \pm 2$.

For $t = x^2 = -2$: $x = \pm\sqrt{2}\, i$.

Answer: $x = 2, -2, \sqrt{2}\, i, -\sqrt{2}\, i$

Problem 3 Solve the following equations:

(1) $x^3 - 9x = 0$ (2) $x^4 = 1$

(3) $x^3 + 5x^2 - 8x - 40 = 0$ (4) $x^4 - 3x^2 + 2 = 0$

1 The Factor Theorem

In this section we will designate integral expressions in x using symbols like $f(x)$ or $g(x)$. Furthermore, let us designate the value of $f(x)$ for $x = 2$ as $f(2)$, and the value of $g(x)$ for $x = -3$ as $g(-3)$ and so on.

Example 1 If we designate integral expression $x^3 - 2x^2 + 3$ as $f(x)$, then we obtain

$$f(-1) = (-1)^3 - 2(-1)^2 + 3 = -1 - 2 + 3 = 0$$

$$f(0) = 3$$

$$f(1) = 1 - 2 + 3 = 2$$

$$f(2) = 2^3 - 2 \bullet 2^2 + 3 = 3.$$

Problem 1 Find the following values for $f(x) = x^3 - 3x + 5$, $g(x) = x^2 + 4$.

(1) $f(-2)$ (2) $g(3)$ (3) $f(-3) + g(4)$

When we divide an integral expression $f(x)$ by a linear expression $x - \alpha$, the remainder is a constant. The following generalization applies to the remainder.

Remainder Theorem

If R is the remainder of an integral expression $f(x)$ divided by $x - \alpha$, then

$$R = f(\alpha).$$

[Proof] If we take $g(x)$ to be the quotient of $f(x)$ divided by $x - \alpha$, then we obtain the following relation:

$$f(x) = (x - \alpha)g(x) + R.$$

Substituting α into this equation, we get

$$f(\alpha) = (\alpha - \alpha)g(\alpha) + R = R.$$

Example 2 Let $f(x) = x^3 - 3x + 4$.

The remainder of $f(x)$ divided by $x - 2$ is

$$f(2) = 2^3 - 3 \cdot 2 + 4 = 6.$$

The remainder of $f(x)$ divided by $x + 3$ is

$$f(-3) = (-3)^3 - 3(-3) + 4 = -14.$$

Problem 2 Find the remainders of $x^3 + 2x - 12$ divided by $x + 1$, $x - 1$, and $x - 2$, respectively.

Problem 3 If we take R to be the remainder of an integral expression $f(x)$ divided by a linear expression $ax + b$, then we obtain

$$R = f(-\frac{b}{a}).$$

Prove this.

Problem 4 Find the remainder when the integral expression $4x^3 - 2x^2 - 9$ is divided by $2x - 1$, $2x + 1$, and $2x - 3$.

Factor Theorem

Suppose $f(x)$ is an integral expression. If α is a solution of the equation $f(x) = 0$, then $x - \alpha$ is a factor of the integral expression $f(x)$.

[Proof] Since α is a solution of the equation $f(x) = 0$, we know that

$$f(\alpha) = 0.$$

If we let R be the remainder of integral expression $f(x)$ divided by $x - \alpha$, then the factor theorem gives us

$$R = f(\alpha) = 0.$$

Therefore, $f(x)$ is divisible by $x - \alpha$. In other words, $x - \alpha$ is a factor of $f(x)$.

Example 3 Let us designate integral expression $x^3 - 2x^2 + 4x - 8$ as $f(x)$. Since $f(2) = 0$, $x - 2$ is a factor of $f(x)$.

Problem 5 Determine whether $x - 3$ is a factor of the following integral expressions. Is $x + 3$ a factor?

(1) $x^3 - 2x^2 - 5x + 6$ (2) $x^4 + x^3 - 8x^2 - 9x - 9$

Demonstration 1 Find the value of p such that $x^3 - 3x^2 + 4x + p$ is divisible by $x - 2$.

[Solution] Let $f(x) = x^3 - 3x^2 + 4x + p$.

Find p such that

$$f(2) = 2^3 - 3 \cdot 2^2 + 4 \cdot 2 + p = 0.$$

We find that

$$p = -4.$$

Problem 6 Find the value of a such that $x^4 - 2x + a$ is divisible by $x + 2$.

Demonstration 2 When an integral expression $f(x)$ is divided by $x - 2$ the remainder is 5, and when it is divided by $x - 3$ the remainder is 9. Find the remainder of $f(x)$ divided by $(x - 2)(x - 3)$.

[Solution] Let $g(x)$ be the quotient of $f(x)$ divided by $(x - 2)(x - 3)$. Since the remainder must be an integral expression of degree one or less, we can express $f(x)$ as:

$$f(x) = (x - 2)(x - 3)g(x) + ax + b \qquad (1)$$

From (1),

substituting 2 for x gives us $f(2) = 2a + b$,

substituting 3 for x gives us $f(3) = 3a + b$.

On the other hand, the remainder theorem tells us that

$$f(2) = 5, \; f(3) = 9.$$

Therefore,

$$2a + b = 5 \tag{2}$$

$$3a + b = 9. \tag{3}$$

Solving (2) and (3), we find

$$a = 4, \; b = -3.$$

Hence, the remainder is $4x - 3$.

Problem 7 Integral expression $f(x)$ is divisible by $x - 4$, and it leaves a remainder of 14 when divided by $x + 3$. Find the remainder when $f(x)$ is divided by $x^2 - x - 12$.

Problem 8 If integral expression $f(x)$ is divisible by both $x - 3$ and $x - 5$, then it is also divisible by $(x - 3)(x - 5)$. Prove this.

5 Solving Equations of Higher Degree Using the Factor Theorem

The factor theorem is often an effective means of solving equations of higher degree, as we can see from the following demonstration.

Demonstration 1

Solve the following equations:

(1) $x^3 + x^2 - 8x - 12 = 0$

(2) $x^4 + x^3 - 6x^2 + x + 3 = 0$

[Solution]

(1) Let $f(x) = x^3 + x^2 - 8x - 12$.

Since $f(-2) = 0, f(x)$ is divisible by $x + 2$ with a quotient of $x^2 - x - 6$. Therefore,

$$f(x) = (x + 2)(x^2 - x - 6).$$

Thus,

$$(x + 2)(x^2 - x - 6) = 0.$$

Hence,

$$x + 2 = 0 \text{ or } x^2 - x - 6 = 0.$$

For $x + 2 = 0$: $x = -2$.

For $x^2 - x - 6 = 0$: $x = -2, 3$.

Answer: $x = -2, -2, 3$

(2) Let $f(x) = x^4 + x^3 - 6x^2 + x + 3$.

Since $f(1) = 0, f(x)$ is divisible by $x - 1$, and the quotient $g(x)$ is

$$g(x) = x^3 + 2x^2 - 4x - 3.$$

Moreover, since $g(-3) = 0, g(x)$ is divisible by $x + 3$ with a quotient of $x^2 - x - 1$. Therefore,

$$f(x) = (x - 1)(x + 3)(x^2 - x - 1).$$

Thus,

$$(x-1)(x+3)(x^2-x-1)=0.$$

Therefore,

$$x-1=0 \text{ or } x+3=0 \text{ or } x^2-x-1=0.$$

Hence,

$$x=1 \text{ or } x=-3 \text{ or } x=\frac{1\pm\sqrt{5}}{2}.$$

Answer: $x = 1, -3, \frac{1\pm\sqrt{5}}{2}$

The left side of equation (1) from Demonstration 1 can be factored into

$$(x+2)^2(x-3)=0.$$

In this case, the solution -2 is referred to as a **double solution**. We gave the solutions of this equation as -2, -2, 3 because we wanted to show that -2 is a double solution. But we can also give them simply as: "Answer: $x = -2, 3$."

Problem 1 Solve the following equations:

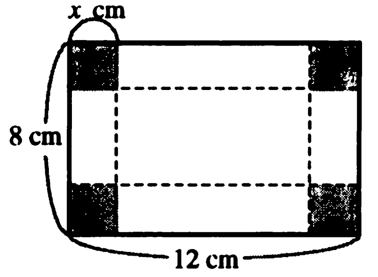
(1) $x^3-4x^2-3x+18=0$ (2) $2x^3-7x^2+7x-2=0$

(3) $6x^3+5x^2-12x+4=0$ (4) $x^4+x^3-x^2+x-2=0$

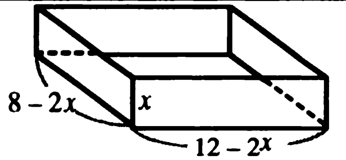
In solving a cubic equation, once we have found one solution, then the remainder of the solution process is equivalent to solving a quadratic equation. Therefore, a cubic equation cannot have more than three solutions. When solving an equation of degree 4, once we have found one solution, then the remainder of the solution process is equivalent to solving a cubic equation. Therefore, an equation of degree 4 cannot have more than four solutions.

We can extend this generalization also to equations of degree 5 and above.

Demonstration 2 Cut out four squares with sides of x cm from four corners of a sheet of construction paper 8 cm wide and 12 cm long, as in the diagram to the right. Then fold it along the dotted lines to make a box with the form of a rectangular parallelepiped. Find the value of x if the volume of this box is 64 cm^3.



[Solution] The condition of the problem tells us that the width of the base of the rectangular parallelepiped is $(8-2x)$ cm, the length is $(12-2x)$ cm, and the height is x cm. Therefore, we can express its volume as



$$x(8-2x)(12-2x) = 64.$$

If we expand and rearrange this equation, we obtain

$$x^3 - 10x^2 + 24x - 16 = 0$$

$$(x-2)(x^2 - 8x + 8) = 0.$$

Hence, $$x = 2 \text{ or } x = 4 \pm 2\sqrt{2}.$$

The condition of the problem specifies that $0 < x < 4$, so

$$x = 2 \text{ or } 4 - 2\sqrt{2}.$$

Problem 2 Find the value of x if the volume of the box in Demonstration 2 is 36 cm^3.

Problem 3 We make a rectangular parallelepiped from a cube by increasing the length by 2 cm and the width by 3 cm, and reducing the height by 1 cm. Then the volume of the rectangular parallelepiped is 60 cm^3. Find the length of a side of the original cube.

Exercises

1. Solve the following sets of simultaneous equations:

(1) $\begin{cases} x + y + z = 8 \\ 3x = 2y + z \\ x - 2y + 3z = -2 \end{cases}$ (2) $\begin{cases} 3x + 2y = 6 \\ 9x^2 - 4y^2 = 108 \end{cases}$

2. Solve the following equations:

(1) $2x^3 - 5x^2 + 4 = 0$ (2) $5x^3 - 12x^2 + x + 6 = 0$

(3) $12x^3 - 8x^2 - 3x + 2 = 0$ (4) $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0$

3. Find the value of k such that the following simultaneous equations have only one pair of solutions. Then find the solutions.

$$\begin{cases} 2x + y = k \\ x^2 + y^2 = 5 \end{cases}$$

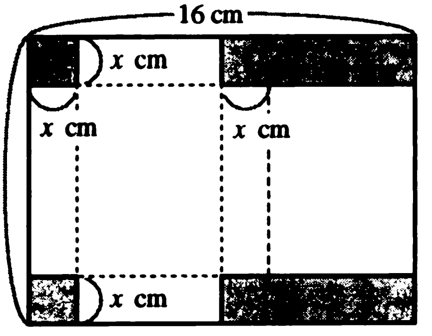
4. Find the value of p and q if the greatest common divisor of the following two expressions is $x - 2$. Then find the least common multiple.

$$x^2 + 2x + p, \quad x^2 + (p + 8)x + q$$

5. Given that $f(x) = x^3 + 2ax^2 - (a + 2b)x - 3b$, find the value of the constants a and b if $f(x)$ is divisible by $(x + 1)(x - 2)$. Then solve the equation $f(x) = 0$.

6. When an integral expression in x, call it $f(x)$, is divided by $x + 4$, it is evenly divisible and the quotient is $g(x)$. Then $g(x)$ is divided by $x - 3$ with a remainder of 5. Find the remainder when $f(x)$ is divided by $x - 3$.

7. We make a box with a lid in the form of a rectangular parallelepiped by cutting out the shaded areas from a rectangular sheet of construction paper 16 cm long and 12 cm wide, as in the figure to the right. If the volume of this box is 96 cm^3, find the value of x.



8. Find the lengths of the two adjacent sides of a rectangle with a perimeter of 42 cm inscribed in a circle of radius 15 cm.

3 INEQUALITIES

1 Solutions of Inequalities

We have already studied comparison of real numbers and the basic properties of inequalities in Chapter 1, Section 1.

We will not consider how we could compare a real number and an imaginary number or two imaginary numbers. Therefore, when working with inequalities, we will assume that all letters represent real numbers.

Linear inequalities in x are easy to solve using the basic properties of inequalities that you learned in junior high school. For example, we can solve

$$3x - 8 > 7 \tag{1}$$

by the following procedure:

Adding 8 to both sides of (1), or transposing -8,

$$3x > 15.$$

Dividing both sides by 3,

$$x > 5. \tag{2}$$

(2) signifies that real number x satisfies inequality (1), in other words, that x is the solution of (1). Therefore, the set of all solutions of (1) is denoted by

$$\{x \mid x > 5\}.$$

We sometimes call "the set of all solutions" simply "the solutions."

Note that when you write the solution to an inequality, you should usually give the answer in the same form as (2).

Demonstration Solve the inequality $-2x + 6 \geq 3x + 16$.

[Solution] Transposing $3x$ to the left side and 6 to the right side, we obtain

$$-5x \geq 10.$$

Dividing both sides by -5, we get

$$x \leq -2.$$

Problem Solve the following inequalities:

(1) $-2x < 5$ (2) $2x - 3 > 5$ (3) $-3x \geq 2$

(4) $x - 2 \leq 5x$ (5) $4(2x - 1) \geq 3(x + 2)$

1 Quadratic Inequalities

Inequalities which can be transformed into one of the forms below by transposing and rearranging are called **quadratic inequalities**.

(quadratic expression) > 0, (quadratic expression) < 0,

(quadratic expression) ≥ 0, (quadratic expression) ≤ 0.

Let's solve the quadratic inequality

$$x^2 + 2x - 8 < 0. \tag{1}$$

Factoring the left side of the inequality, we get

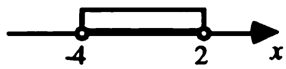
$$x^2 + 2x - 8 = (x + 4)(x - 2).$$

Then by inspecting the sign of the left side for various values of x, we obtain the following table.

x	$x < -4$	-4	$-4 < x < 2$	2	$2 < x$
$x + 4$	$-$	0	+	+	+
$x - 2$	$-$	$-$	$-$	0	+
$(x + 4)(x - 2)$	+	0	$-$	0	+

Inequality (1) is satisfied by x if x satisfies $-4 < x < 2$, as you can see from the table.

Therefore, the solution of inequality (1) is



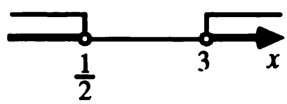
$$-4 < x < 2.$$

Demonstration 1 Solve the inequality $2x^2 - 7x + 3 > 0$.

[Solution] If we factor the left side, we obtain $(2x - 1)(x - 3)$. Inspecting the signs gives us the following table.

x	$x < \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} < x < 3$	3	$3 < x$
$2x - 1$	$-$	0	$+$	$+$	$+$
$x - 3$	$-$	$-$	$-$	0	$+$
$(2x - 1)(x - 3)$	$+$	0	$-$	0	$+$

From this table the solutions are



$$x < \frac{1}{2},\ 3 < x.$$

Problem 1 Solve the following inequalities:

(1) $x^2 - 4x > 0$ (2) $x^2 - 8x + 12 \le 0$

(3) $2x^2 - 3x - 2 < 0$ (4) $3x^2 - 7x + 10 \ge 2x^2$

(5) $(x + 1)(x + 4) > 18$ (6) $\frac{5x^2}{2} - \frac{x}{6} \ge \frac{1}{3}$

If a quadratic equation $ax^2 + bx + c = 0$ has two non-equal real solutions α and β, then it can be factored into

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

For $a > 0$, the sign of $ax^2 + bx + c$ matches the sign of $(x - \alpha)(x - \beta)$. If we let $\alpha < \beta$, then by inspecting the sign of

$$(x - \alpha)(x - \beta)$$

we can compile the following table.

x	$x < \alpha$	α	$\alpha < x < \beta$	β	$\beta < x$
$x - \alpha$	$-$	0	$+$	$+$	$+$
$x - \beta$	$-$	$-$	$-$	0	$+$
$(x-\alpha)(x-\beta)$	$+$	0	$-$	0	$+$

We can now formulate the following general rules for solving quadratic inequalities.

Solutions of Quadratic Inequalities for $D > 0$

For $a > 0$, if a quadratic equation $ax^2 + bx + c = 0$ has two non-equal real roots α and β, and $\alpha < \beta$,

(1) the solutions of $ax^2 + bx + c > 0$ are

$$x < \alpha,\ \beta < x.$$

(2) the solutions of $ax^2 + bx + c < 0$ are

$$\alpha < x < \beta.$$

(3) the solutions of $ax^2 + bx + c \geq 0$ are

$$x \leq \alpha,\ \beta \leq x.$$

(4) the solutions of $ax^2 + bx + c \leq 0$ are

$$\alpha \leq x \leq \beta.$$

Demonstration 2 Solve the inequality $1 + 2x - x^2 \geq 0$.

[Solution] Multiplying both sides by -1 to make the coefficient of x^2 positive, we obtain

$$x^2 - 2x - 1 \leq 0.$$

Solving the equation $x^2 - 2x - 1 = 0$, we get

$$x = 1 \pm \sqrt{2}.$$

Therefore, the solution of the inequality is

$$1 - \sqrt{2} \leq x \leq 1 + \sqrt{2}.$$

Problem 2 Solve the following inequalities:

(1) $x^2 - 4x - 1 > 0$ (2) $2x^2 - 6x + 3 < 0$

(3) $-5x^2 - 3x + 1 \leq 0$ (4) $5 + 8x + 2x^2 \leq 0$

Problem 3 Find the value of p and q if the solutions of the quadratic inequality $x^2 + px + q > 0$ are $x < -3, 2 < x$.

Next let's consider the case when a quadratic equation $ax^2 + bx + c = 0$ has a multiple solution α. Now, we have

$$ax^2 + bx + c = a(x - \alpha)^2.$$

For $a > 0$, the inequality $a(x - \alpha)^2 \geq 0$ always holds. Only when $x = \alpha$ do we have $a(x - \alpha)^2 = 0$.

We can now formulate the following generalization:

Solutions of Quadratic Inequalities for $D = 0$

For $a > 0$, if quadratic equation $ax^2 + bx + c = 0$ has a multiple solution α,

(1) the solution of $ax^2 + bx + c > 0$ is all real numbers except α.

(2) $ax^2 + bx + c < 0$ has no solution.

(3) the solution of $ax^2 + bx + c \geq 0$ is all real numbers.

(4) the solution of $ax^2 + bx + c \leq 0$ is $x = \alpha$.

Demonstration 3 Solve the inequality $16x^2 - 8x + 1 > 0$.

[Solution] Solving the equality $16x^2 - 8x + 1 = 0$, we obtain

$(4x - 1)^2 = 0$, and hence $x = \frac{1}{4}$.

This is a multiple solution.

The coefficient of x^2 is positive.

Therefore, the solution of this inequality is all real numbers except $\frac{1}{4}$.

Note: You can also express the solution of the above Demonstration simply as $x \neq \frac{1}{4}$ or $x < \frac{1}{4}, \frac{1}{4} < x$.

Problem 4 Solve the following inequalities:

(1) $x^2 + 4x + 4 > 0$ (2) $4x^2 - 4x + 1 < 0$

(3) $9x^2 - 12x + 4 \geq 0$ (4) $64 \leq 16x - x^2$

Finally, let's consider the case when a quadratic equation $ax^2 + bx + c = 0$ has no real solutions, or has imaginary solutions.

If we replace the discriminant $b^2 - 4ac$ with D in the above equation, then

$$\begin{aligned} ax^2 + bx + c &= a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} \\ &= a(x + \frac{b}{2a})^2 + (\frac{-D}{4a}). \end{aligned}$$

Since the equation has imaginary solutions, that is, $D < 0$, for $a > 0$ we have $\frac{-D}{4a} > 0$. The inequality $ax^2 + bx + c > 0$ always holds true, regardless of what real number we substitute for x.

Therefore we can formulate the following generalization:

Solutions of Quadratic Inequalities for $D < 0$

For $a > 0$, if the quadratic equation $ax^2 + bx + c = 0$ has imaginary solutions,

(1) the solution of $ax^2 + bx + c > 0$ is all real numbers.

(2) $ax^2 + bx + c < 0$ has no solution.

(3) the solution of $ax^2 + bx + c \geq 0$ is all real numbers.

(4) $ax^2 + bx + c \leq 0$ has no solution.

Demonstration 4 Solve the inequality $2x^2 + 5x + 7 > 0$.

[Solution]

Let D be the discriminant of the equation $2x^2 + 5x + 7 = 0$:

$$D = 25 - 56 = -31 < 0.$$

And the coefficient of x^2 is positive.

Therefore, the solution of this inequality is the set of all real numbers.

Problem 5 Solve the following inequalities:

(1) $x^2 - x + 1 > 0$ (2) $x^2 + 2x + 5 < 0$

(3) $2x^2 - 4x + 3 \geq 0$ (4) $5x \geq x^2 + 9$

Demonstration 5 Find the range of x such that it satisfies both of the following inequalities at the same time.

$$\begin{cases} 3x^2 - 11x - 4 < 0 & (1) \\ 2x^2 + x - 1 \geq 0 & (2) \end{cases}$$

[Solution] Since $3x^2 - 11x - 4 = (3x + 1)(x - 4)$,

the solution of (1) is

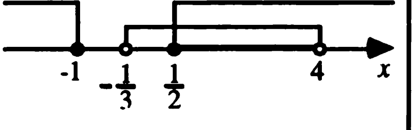
$$-\frac{1}{3} < x < 4. \qquad (3)$$

Since $2x^2 + x - 1 = (x + 1)(2x - 1)$,

the solution of (2) is

$$x \leq -1,\ \frac{1}{2} \leq x. \qquad (4)$$

Mark (3) and (4) on a number line and note the common range. Then the range of x we want to find is



$$\frac{1}{2} \leq x < 4.$$

Problem 6 Find the range of x which satisfies the following two inequalities at the same time.

$$\begin{cases} x^2 - 6x + 5 \leq 0 \\ x^2 - 9x + 14 < 0 \end{cases}$$

Problem 7 Find the range of x that satisfies $2x + 5 < x^2 < 8 + 7x$.

Problem 8 Find the integer value of x that satisfies the two inequalities $x^2 - 4x + 2 > 0$ and $x^2 + 2x - 15 < 0$ at the same time.

Demonstration 6 Find the range of real numbers k such that the quadratic equation

$$x^2 + 2kx - 3k - 2 = 0$$

has two non-equal real solutions.

[Solution] Let D be the discriminant of this equation.

$$\frac{D}{4} = k^2 - (-3k - 2) = k^2 + 3k + 2$$

Therefore, the necessary and sufficient condition for this equation to have two non-equal real solutions is

$$k^2 + 3k + 2 > 0$$

$$(k + 2)(k + 1) > 0.$$

Hence, $k < -2,\ -1 < k$.

Problem 9 Find the range of real numbers p such that the quadratic equation $x^2 + 2(2p + 1)x - (p^2 - 1) = 0$ has imaginary solutions.

Exercises

1. Solve the following inequalities:

(1) $x^2 - 7x + 12 > 0$

(2) $x^2 + 4x - 7 \leq 0$

(3) $x^2 - 8x + 3 \geq 0$

(4) $x^2 - 12x + 36 \leq 0$

(5) $2x^2 - 3x + 6 > 0$

(6) $2x^2 + 10x + 18 \leq 0$

(7) $x^2 + 5x - 3 > 3x^2 - 2x$

(8) $3x^2 - 6x + 1 < 2x^2 - 17$

(9) $1 < x(x + 8)$

(10) $x^2 + 3x \leq \frac{3}{4}x^2 - \frac{1}{4}x - \frac{15}{2}$

2. The perimeter of a rectangle is 40 cm and the area is 64 cm^2 or more and 91 cm^2 or less. Find the range of the width of this rectangle.

3. Given the following two sets M and N:

$$M = \{x \mid 6x^2 - 7x - 5 > 0\}$$

$$N = \{x \mid x^2 - 2x - 8 < 0\}.$$

Find the integer number from among the elements of $M \cap N$.

4. Given the quadratic equation $x^2 + 2(m - 4)x + 3m - 2 = 0$. Find the range of real numbers m such that this equation has real solutions.

5. Find the solution of the quadratic inequality $x^2 - (a + 2)x + 2a < 0$ for $a > 2$.

6. Find the values of p and q such that the quadratic inequality $2x^2 + px + q < 0$ has the solution $\frac{1}{2} < x < 4$.

4 EXPRESSIONS AND PROOFS

1 Proving Equalities

The equality

$$(a-b)(a+b) = a^2 - b^2 \tag{1}$$

can be proved in the following way, based on the basic law of arithmetic operations:

$$\begin{aligned}(a-b)(a+b) &= a(a+b) - b(a+b)\\ &= a^2 + ab - ba - b^2\\ &= a^2 + ab - ab - b^2 = a^2 - b^2.\end{aligned}$$

Equality (1) always holds true, regardless of what numbers we substitute for a and b. Such an equality is called an **identity**.

Demonstration 1 Prove the following equality.

$$(ac+bd)^2 + (ad-bc)^2 = (a^2+b^2)(c^2+d^2)$$

[Proof]

$$\begin{aligned}\text{Left side } &= (a^2c^2 + 2abcd + b^2d^2) + (a^2d^2 - 2abcd + b^2c^2)\\ &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2\\ &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2.\end{aligned}$$

Therefore, left side = right side.

Note: The instruction "Prove the equality" means to prove that the equality is an identity.

Problem 1 Prove the following equalities:

(1) $(a^2 + 3b^2)(c^2 + 3d^2) = (ac + 3bd)^2 + 3(ad - bc)^2$

(2) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$

(3) $a^2 + b^2 + c^2 - bc - ca - ab = \frac{1}{2}\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$

Demonstration 2 Prove the following equality:

$$\frac{1}{1-x} + \frac{1}{1-y} = 1 + \frac{1-xy}{(1-x)(1-y)}.$$

[Proof]

Left side

$$= \frac{1-y}{(1-x)(1-y)} + \frac{1-x}{(1-x)(1-y)} = \frac{2-x-y}{(1-x)(1-y)}.$$

right side

$$= \frac{(1-x)(1-y) + 1 - xy}{(1-x)(1-y)} = \frac{2-x-y}{(1-x)(1-y)}.$$

Therefore, left side = right side.

Note: When we prove equalities involving fractional expressions, we should omit the case when the denominator is equal to 0.

Problem 2 Prove the following equality:

$$\frac{b}{a(a+b)} + \frac{c}{(a+b)(a+b+c)} = \frac{1}{a} - \frac{1}{a+b+c}.$$

In order to determine whether or not a particular equality is an identity, we must sometimes note that only certain letters can be variables. For example, if a certain equality is an identity for x, then that equality always holds true regardless of what numbers are substituted for x.

For an integral expression of the form $f(x) = ax^3 + bx^2 + cx + d$, let's consider under what conditions the following equality is an identity for x:

$$f(x) = ax^3 + bx^2 + cx + d = 0. \qquad (1)$$

If (1) is an identity, then (1) holds true for any value of x. Let's substitute 0, 1, -1, 2 for x.

$$f(0) = d = 0$$

$$f(1) = a + b + c + d = a + b + c = 0 \tag{2}$$

$$f(-1) = -a + b - c + d = -a + b - c = 0 \tag{3}$$

$$f(2) = 8a + 4b + 2c + d = 8a + 4b + 2c = 0 \tag{4}$$

From (2) and (3), $\qquad b = 0,\ a + c = 0.$

Substituting $b = 0, c = -a$ into (4), we get $6a = 0$.

Therefore, $\qquad a = 0,\ c = 0$

Hence, if (1) is an identity, we obtain

$$a = b = c = d = 0. \tag{5}$$

Conversely, if (5) holds, then (1) holds for all x.

In other words, the necessary and sufficient condition for (1) to be an identity is that all the coefficients must be equal to 0.

Problem 3 Find the value of a, b, and c such that the following equality is an identity for x.

$$(a - 2b)x^2 + (a + b + c)x + (a - b - 1) = 0$$

The fact that the following equality is an identity for x,

$$ax^3 + bx^2 + cx + d = a'x^3 + b'x^2 + c'x + d' \tag{6}$$

means that the equation obtained by transposing the right side to the left side is also an identity:

$$(a - a')x^3 + (b - b')x^2 + (c - c')x + (d - d') = 0.$$

Therefore, the necessary and sufficient condition for (6) to be an identity is

$$a - a' = b - b' = c - c' = d - d' = 0.$$

That is,

$$a = a', b = b', c = c', d = d'.$$

In general, given two integral expressions $f(x)$ and $g(x)$,

$$f(x) = g(x)$$

is an identity if and only if all the coefficients of the same degree in both $f(x)$ and $g(x)$ are the same.

Example $ax^3 + 2x^2 + bx + 3 = 4x^3 + cx^2 - 5x + 3$

is an identity if and only if $a = 4$, $b = -5$, and $c = 2$.

Demonstration 3 Find the value of the constants a, b, and c such that the following equality is an identity.

$$\frac{5x + 2}{x^3 + 1} = \frac{a}{x + 1} + \frac{bx + c}{x^2 - x + 1}$$

[Solution]

Reducing the right side to a common denominator, we obtain

$$\frac{a}{x + 1} + \frac{bx + c}{x^2 - x + 1} = \frac{a(x^2 - x + 1) + (x + 1)(bx + c)}{x^3 + 1}.$$

Since this denominator is the same as the denominator of the left side, it is sufficient to show that

$$5x + 2 = a(x^2 - x + 1) + (x + 1)(bx + c)$$

is an identity in order to prove that the given equality is also an identity.

Rearranging the right side,

$$5x + 2 = (a + b)x^2 + (-a + b + c)x + (a + c).$$

Therefore,

$$a + b = 0, \qquad -a + b + c = 5, \qquad a + c = 2.$$

Hence the values we are looking for are

$$a = -1, b = 1, c = 3.$$

Problem 4 Find the values of the constants a, b, and c such that the following equalities are identities for x.

(1) $(3x + 1)(x^2 + ax + b) = 3x^3 + cx^2 + x + (3a - c)$

(2) $a(x - 1)(x - 2) + b(x - 2)(x - 3) + c(x - 3)(x - 1) = 3x + 5$

(3) $\dfrac{3x + 4}{x(x^2 + 2)} = \dfrac{a}{x} + \dfrac{bx + c}{x^2 + 2}$

(4) $\dfrac{3x^2 - 4x + 2}{x(x - 1)^2} = \dfrac{a}{x} + \dfrac{b}{x - 1} + \dfrac{c}{(x - 1)^2}$

Proving Equalities for a Given Condition

There are some equalities which are not identities, but do hold for a certain given condition.

Demonstration 4 Show that if $x + y = 1$, then $x^2 - x = y^2 - y$.

[Proof]

$$(x^2 - x) - (y^2 - y) = (x^2 - y^2) - (x - y)$$
$$= (x - y)(x + y - 1)$$

Since $x + y - 1 = 0$,

$$(x^2 - x) - (y^2 - y) = 0.$$

Hence, $x^2 - x = y^2 - y$.

Demonstration 5 Prove the following equality for $a + b + c = 0$:

$$2a^2 + bc = (b - a)(c - a).$$

[Proof]

From the given condition

$$c = -(a + b).$$

Rewriting both sides of the equality that we want to prove using this condition, we obtain

$$\text{left side} = 2a^2 - b(a + b) = 2a^2 - ab - b^2$$

$$\text{right side} = (b - a)(-b - 2a) = 2a^2 - ab - b^2.$$

Therefore, left side = right side.

Problem 5 Prove the equality in Demonstration 5 by factoring the expression obtained by subtracting the right side from the left side, just as in Demonstration 4.

Problem 6 Prove the following equalities for $a + b + c = 0$:

(1) $a^2 - bc = b^2 - ca = c^2 - ab$

(2) $\dfrac{b^2 - c^2}{a} + \dfrac{c^2 - a^2}{b} + \dfrac{a^2 - b^2}{c} = 0$

Proportional Expressions

Demonstration 6 Prove the following equalities for $\frac{a}{b} = \frac{c}{d}$.

(1) $$\frac{a+b}{b} = \frac{c+d}{d}$$

(2) $$\frac{a+2b}{2a+b} = \frac{c+2d}{2c+d}$$

[Proof] If we let $\frac{a}{b} = \frac{c}{d} = k$, then $a = bk, c = dk$.

(1) $$\frac{a+b}{b} = \frac{bk+b}{b} = k+1$$

$$\frac{c+d}{d} = \frac{dk+d}{d} = k+1$$

Therefore, $\frac{a+b}{b} = \frac{c+d}{d}$.

(2) $$\frac{a+2b}{2a+b} = \frac{(k+2)b}{(2k+1)b} = \frac{k+2}{2k+1}$$

$$\frac{c+2d}{2c+d} = \frac{(k+2)d}{(2k+1)d} = \frac{k+2}{2k+1}$$

Therefore, $\frac{a+2b}{2a+b} = \frac{c+2d}{2c+d}$.

Problem 7 Prove the following equalities for $\frac{a}{b} = \frac{c}{d}$.

(1) $\frac{a-b}{b} = \frac{c-d}{d}$ (2) $\frac{(a+b)^2}{ab} = \frac{(c+d)^2}{cd}$

We can rewrite

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

as

$$a : b : c = a' : b' : c'.$$

We read $a : b : c$ as "a is to b is to c," and refer to it as an **extended ratio**.

Demonstration 7 Show that if $x : y : z = 4 : 3 : 2$, then

$$x + 2y : 2x - y : 5z = 2 : 1 : 2.$$

[Proof] If we let $\frac{x}{4} = \frac{y}{3} = \frac{z}{2} = k$, then

$$x = 4k, \ y = 3k, \ z = 2k.$$

Therefore,

$$\frac{x + 2y}{2} = \frac{4k + 6k}{2} = 5k$$

$$\frac{2x - y}{1} = 8k - 3k = 5k$$

$$\frac{5z}{2} = \frac{10k}{2} = 5k.$$

Hence, $$\frac{x + 2y}{2} = \frac{2x - y}{1} = \frac{5z}{2}.$$

So, $x + 2y : 2x - y : 5z = 2 : 1 : 2.$

Problem 8 Demonstrate that if $x : y : z = 2 : 3 : 4$, then $x^2 : y^2 : z^2 = 4 : 9 : 16$.

Problem 9 Prove that for $a : b : c = a' : b' : c'$,

$$(a + b + c) : (a' + b' + c') = a : a'.$$

2 Proving Inequalities

Demonstration 1

(1) Prove the following inequality:

$$a^2 + b^2 \geq ab.$$

(2) Prove that the equality portion of the above inequality holds if and only if $a = b = 0$.

[Proof]

(1) Left side – right side $= a^2 - ab + b^2$

$$= (a - \frac{b}{2})^2 + \frac{3}{4}b^2.$$

Since a and b are real numbers,

$$(a - \frac{b}{2})^2 \geq 0, \ \frac{3}{4}b^2 \geq 0.$$

Therefore, $a^2 - ab + b^2 \geq 0$.

So, $a^2 + b^2 \geq ab$.

(2) If $a^2 + b^2 = ab$, then $(a - \frac{b}{2})^2 + \frac{3}{4}b^2 = 0$.

Hence, $a - \frac{b}{2} = 0$ and $b = 0$.

Therefore, $a = b = 0$.

Note: The instruction "Prove the inequality" means to prove that the inequality holds for any real numbers a and b.

Problem 1 Prove the following inequalities, and determine when the equalities hold.

(1) $a^2 + ab + b^2 \geq 0$

(2) $5x^2 - 4xy + 6y^2 \geq 0$

(3) $x^2 + y^2 \geq 4x - 6y - 13$

Problem 2 Prove the following inequality:

$$(a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2.$$

For $a > 0, b > 0$

$$a > b \Leftrightarrow a^2 > b^2.$$

[Proof] It is sufficient to prove that

$$a - b > 0 \Leftrightarrow (a - b)(a + b) > 0.$$

Since $a > 0$ and $b > 0$, $a + b > 0$.

Hence, the signs of $a - b$ and $(a - b)(a + b)$ are the same.

Therefore,

$$a - b > 0 \Leftrightarrow (a - b)(a + b) > 0.$$

Demonstration 2 Prove the following inequality for $a > 0, b > 0$:

$$\sqrt{a} + \sqrt{b} > \sqrt{a + b}\ .$$

[Proof] Since both sides of this inequality are positive, the ratio of their magnitudes will not change if both sides are squared.

$$(\text{left side})^2 - (\text{right side})^2 = (\sqrt{a} + \sqrt{b})^2 - (a + b)$$

$$= a + 2\sqrt{ab} + b - (a + b) = 2\sqrt{ab} > 0$$

Therefore,

$$(\sqrt{a} + \sqrt{b})^2 > (\sqrt{a + b})^2.$$

So,

$$\sqrt{a} + \sqrt{b} > \sqrt{a + b}\ .$$

Problem 3 Prove the following inequality for $a > b > 0$:

$$\sqrt{a} - \sqrt{b} < \sqrt{a-b}.$$

Arithmetic Mean and Geometric Mean

For $a \geq 0, b \geq 0$,

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

This equality holds if and only if $a = b$.

[Proof] If we let $\sqrt{a} = x, \sqrt{b} = y$, then

$$a = x^2, \quad b = y^2, \quad \sqrt{ab} = xy.$$

Therefore, $\dfrac{a+b}{2} - \sqrt{ab} = \dfrac{x^2 + y^2}{2} - xy$

$$= \frac{(x-y)^2}{2} \geq 0.$$

Hence, $$\frac{a+b}{2} \geq \sqrt{ab}.$$

Moreover, $\dfrac{a+b}{2} = \sqrt{ab}$ holds when $\dfrac{(x-y)^2}{2} = 0$, and therefore when $x = y$ or $a = b$.

Note: $\dfrac{a+b}{2}$ and $\sqrt{ab}$ are referred to as the **arithmetic mean** and the **geometric mean** of a and b.

Demonstration 3 Prove the following inequality for $a > 0, b > 0$. Then determine when the equality holds.

$$(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$$

[Proof]

From the relation of the arithmetic and geometric means,

$$a + b \geq 2\sqrt{ab} \qquad (1)$$

$$\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}. \qquad (2)$$

Since both sides of (1) and (2) are positive, we can multiply one right side by the other right side and one left side by the other left side, and then we get

$$(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4.$$

The equality holds when the equalities of (1) and (2) both hold, that is, when $a = b$.

Problem 4 Prove the following inequality, assuming that all the letters represent positive numbers. Determine when the equality holds.

(1) $a + \frac{1}{a} \geq 2$

(2) $\left(\frac{a}{b} + \frac{c}{d}\right)\left(\frac{b}{a} + \frac{d}{c}\right) \geq 4$

Problem 5 Prove the following inequality, assuming that $x > 0$ and $y > 0$.

$$\sqrt{x} + \sqrt{y} \leq \sqrt{2(x + y)}$$

Demonstration 4 Prove $|x + y| \leq |x| + |y|$.

[Proof]

Since $x \leq |x|$ and $y \leq |y|$,

$$x + y \leq |x| + |y|. \qquad (1)$$

And since $-x \leq |x|$ and $-y \leq |y|$,

$$-(x + y) \leq |x| + |y|. \qquad (2)$$

As $|x + y|$ is either $x + y$ or $-(x + y)$, (1) and (2) give us

$$|x + y| \leq |x| + |y|.$$

Problem 6 Prove the inequality in Demonstration 4 by comparing the magnitude of $|x+y|^2$ and $(|x|+|y|)^2$.

Problem 7 Prove $|x|-|y| \leq |x+y|$.

2 The Contrapositive and Indirect Proof

As we learned on page 70, the implication $a=0 \Rightarrow ab=0$ holds, but the converse $ab=0 \Rightarrow a=0$ does not hold.

> Even if $p \Rightarrow q$ holds, the converse $q \Rightarrow p$ does not always hold.

Problem 1 State the converse of the following implications, and determine whether or not they are true:

(1) $x=1 \Rightarrow x^2-2x+1=0$.

(2) $x=2 \Rightarrow x^2-5x+6=0$.

(3) $x>1 \Rightarrow x^2-1>0$.

(4) Any multiple of 6 is divisible by 3.

Problem 2 Examine the converse of the implication $a=b=0 \Rightarrow a^2+b^2=0$ for the following condition.

(1) a and b are real numbers.

(2) a and b are complex numbers.

An assertion that p does not hold, in other words, the condition that can be referred to as "not p," is called the **negation** of p. We will represent the negation of p as $\overline{p}$.

Example 1 Given an integer n, the negation of an assertion that n is an odd number means that n is an even number.

Example 2 Given a real number x, the negation of $x > 1$ is $x \leq 1$. The negation of $x \leq -3$ is $x > -3$.

Example 3 If p is "$x > 0$ or $y > 0$," then $\bar{p}$ is

"$x \leq 0$ and $y \leq 0$."

If p is "$x > 0$ and $y > 0$," then $\bar{p}$ is

"$x \leq 0$ or $y \leq 0$."

Note: We usually write the statement "$x > 0$ and $y > 0$" simply as "$x > 0, y > 0$."

Problem 3 State the negation of the following conditions:

(1) $x \leq 4$ (2) $-1 < a < 2$

(3) At least one of a and b is equal to 0.

Indirect Proof

Demonstration 1 If a square of a certain integer is an even number, then the original integer must be an even number. Prove this.

[Proof] Let n be a certain integer and n^2 its square.

To show that n is an even number, it is sufficient to show that n is not an odd number. To do so, we need only deduce a contradiction by assuming that n is an odd number.

If we assume that n is an odd number, then n can be expressed as

$$n = 2k + 1$$

for a certain integer k. Therefore,

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1.$$

Since $4k^2 + 4k$ is an even number, $4k^2 + 4k + 1$ must be an odd number. Therefore, n^2 is an odd number. This fact contradicts the assumption that "n^2 is an even number."

Hence n is not an odd number. Equivalently, n is an even number.

Problem 4 Prove that a number whose square is an odd number is itself also an odd number, by the same method as above.

We can prove a certain conclusion by

"assuming the negation of that conclusion and deducing a contradiction."

This method is called **indirect proof**.

In Demonstration 1, we wanted to prove that

$$(n^2 \text{ is an even number}) \Rightarrow (n \text{ is an even number}),$$

but we actually proved that

$$(n \text{ is an odd number}) \Rightarrow (n^2 \text{ is an odd number}).$$

In general, if we want to prove $p \Rightarrow q$, we can also simply prove

$$\bar{q} \Rightarrow \bar{p}.$$

$\bar{q} \Rightarrow \bar{p}$ is called **the contrapositive** of $p \Rightarrow q$. The contrapositive of $\bar{q} \Rightarrow \bar{p}$ is the original $p \Rightarrow q$.

> If the contrapositive $\bar{q} \Rightarrow \bar{p}$ is true, then $p \Rightarrow q$ is also true.
>
> Moreover, if $p \Rightarrow q$ is true, then the contrapositive $\bar{q} \Rightarrow \bar{p}$ is also true.

Demonstration 2 Prove that if $ab < 0$, then "$a > 0$ or $b > 0$."

[Proof] We want to prove that

$$ab < 0 \Rightarrow a > 0 \text{ or } b > 0.$$

The contrapositive of this implication is

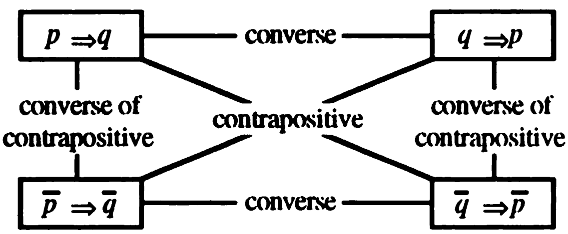
$$a \leq 0, b \leq 0 \Rightarrow ab \geq 0.$$

It is immediately clear that this implication is true.

Problem 5 Prove that if $a^2 + b^2 \leq 4$, then "$a \leq \sqrt{2}$ or $b \leq \sqrt{2}$."

Problem 6 Prove that if $x + \dfrac{1}{x} < 2$, then $x < 0$.

$\bar{p} \Rightarrow \bar{q}$ is called the **converse of the contrapositive** of $p \Rightarrow q$. The relations among the converse, the contrapositive, and the converse of the contrapositive can be summarized as in the diagram below.



Proof That $\sqrt{2}$ Is an Irrational Number

To prove that $\sqrt{2}$ is an irrational number, it is sufficient to deduce a contradiction by assuming that $\sqrt{2}$ is a rational number.

If we assume that $\sqrt{2}$ is a rational number, then we can express $\sqrt{2}$ as an irreducible fraction with integers m and n as numerator and denominator.

$$\sqrt{2} = \frac{m}{n} \qquad (1)$$

Since the right side is irreducible, the greatest common divisor of m and n is 1.

Multiplying both sides by n and then squaring both sides, we obtain

$$2n^2 = m^2. \tag{2}$$

Then m^2 is an even number. Therefore, from Demonstration 1 on page 115 we know that m is an even number. Hence, m can be expressed as

$$m = 2k \tag{3}$$

for a certain integer k. If we now substitute (3) into (2) and divide both sides by 2, we get

$$n^2 = 2k^2.$$

Therefore, n^2 is an even number, and so n is also an even number. Here, m and n are both even numbers, and we can show that they have 2 as their common divisor. This fact contradicts the original assumption that $\frac{m}{n}$ is an irreducible fraction.

Thus $\sqrt{2}$ is not a rational number. In other words, $\sqrt{2}$ is an irrational number.

Problem 7 Prove the following statements:

(1) If an integer n is not a multiple of 3, then n can be expressed as $n = 3k + 1$ or $n = 3k + 2$ for a certain integer k.

(2) Given any integer n, if n^2 is a multiple of 3, then n is a multiple of 3.

Problem 8 Prove that $\sqrt{3}$ is an irrational number using (2) from Problem 7.

Exercises

1. Find the value of constants a, b, and c such that the equality $x^2 + axy + y^2 + bx + cy + 1 = 0$ always holds for $x + y = 1$.

2. Given $a + b + c = 0$:

   (1) Prove that $a^3 + b^3 + c^3 = 3abc$.

   (2) Prove the following equality using the result of (1).

$$\frac{a^2 + b^2 + c^2}{a^3 + b^3 + c^3} + \frac{2}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$$

3. Prove that $(a + b + c)(a - b + c) = a^2 + b^2 + c^2$ for $\frac{b}{a} = \frac{c}{b}$.

4. Prove the following inequalities for $x > 0$, $y > 0$. Determine when the equalities hold.

   (1) $\sqrt{xy} \geq \frac{2xy}{x + y}$ (2) $\frac{1}{2x} + \frac{1}{2y} \geq \frac{2}{x + y}$

5. Prove the following inequality for positive numbers a, b, c, and d, where $\frac{a}{b} < \frac{c}{d}$.

$$\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$$

6. State the converse and contrapositive of the following implications. Determine whether or not the converse is true.

   (1) If $a = b$, then $a^2 = b^2$.

   (2) If $x > 1$ and $y > 1$, then $x + y > 2$.

   (3) If $x < 2$ or $3 < x$, then $x^2 - 5x + 6 > 0$.

7. Use indirect proof to show that for positive numbers a and b if $a^2 + b^2 > 50$, then either a or b must be greater than 5.

Chapter Exercises

A

1. Solve the following equations:

(1) $(x-1)^2 + 3(x-1) + 2 = 0$ (2) $(x-1)(x-2)(x-3) = 24$

(3) $(x^2 - x)^2 - 4(x^2 - x) - 12 = 0$

2. Solve the following simultaneous equations:

(1) $\begin{cases} x - y = 2 \\ 2y = z + x \\ 4x + 3y - 5z = 24 \end{cases}$ (2) $\begin{cases} x - 2y = 3 \\ x^2 - 4xy + y^2 = 6 \end{cases}$

3. Find the value of real numbers a and b if the following equalities hold.

(1) $(a+b) + abi = 3 + i$ (2) $2 + (a + b^2)i = a - b + 4i$

4. If α and β are two solutions of quadratic equation $ax^2 + bx + c = 0$, and D is the discriminant, then $a^2(\alpha - \beta)^2 = \mathrm{D}$. Prove this.

5. Find the values of a and b if $f(x) = 4x^3 + ax^2 + bx - 25$ is divisible by $2x^2 - 3x - 5$.

6. Prove the following implications if we let $\dfrac{b+c}{a} = \dfrac{c+a}{b} = \dfrac{a+b}{c} = k$.

(1) If $a + b + c = 0$, then $k = -1$.

(2) If $a + b + c \neq 0$, then $k = 2$.

7. Prove that for real numbers a and b, "$a + b > 0$, $ab > 0$" is the necessary and sufficient condition for "$a > 0$, $b > 0$."

8. Prove the inequality $|a| + |b| \leq \sqrt{2(a^2 + b^2)}$.

9. Prove the following statement by indirect proof.

For natural numbers a, b, and c, if $a^2 + b^2 = c^2$, then at least one of these three numbers is even.

B

1. Solve the following equations:

(1) $(x^2 - 6x + 7)(x^2 - 6x + 6) = 2$

(2) $(x - 3)(x - 1)(x + 2)(x + 4) + 24 = 0$

2. Factor $x^4 + x^2 - 6$ in the realm of:

(1) rational numbers;

(2) real numbers;

(3) complex numbers.

3. Let α and β be two solutions of a quadratic equation $x^2 - ax + b = 0$. Find the values of a and b if the solutions of the quadratic equation $x^2 - 5x + 8 = 0$ are $\alpha + 1$ and $\beta + 1$.

4. Given real number k, let α and β be two solutions of the quadratic equation $x^2 - kx - 5 = 0$. Find the value of k such that the value of $|\alpha - \beta|$ is a minimum.

5. Find the value of the following expression for $xyz = 1$:

$$\frac{x}{xy + x + 1} + \frac{y}{yz + y + 1} + \frac{z}{zx + z + 1}$$

6. Prove the following equality for $a : b : c = x : y : z$.

$$(a^2 + b^2 + c^2) : (x^2 + y^2 + z^2) = (ab + bc + ca) : (xy + yz + zx)$$

7. Prove the following inequality for positive numbers $a, b, m,$ and n with $m + n = 1$.

$$\sqrt{ma + nb} \geq m\sqrt{a} + n\sqrt{b}$$

8. What kind of condition is A in order for B to hold in (1) – (3): necessary, sufficient, or necessary and sufficient?

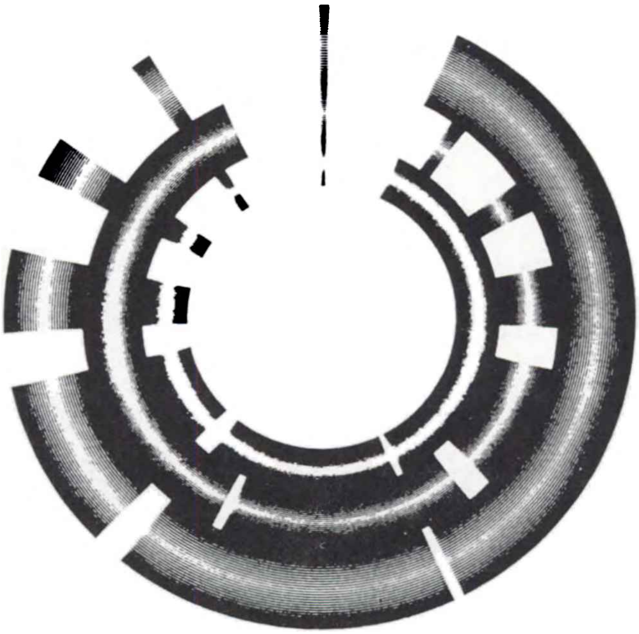
   (1) $A : x \geq 0;\ B : x = \dfrac{x + |x|}{2}$.

   (2) $A : ab \neq 0;\ B : |a| + |b| > |a + b|$.

   (3) A : quadratic equation $x^2 - px - q = 0$ has imaginary solutions;
   $B : p + q < 1$.

CHAPTER 3

PLANE FIGURES AND EXPRESSIONS



Descartes (1596 – 1650) was the first to propose using coordinates to study the properties of figures. This method requires that we first write an expression for a relation between figures, and then perform algebraic, differential, and integral operations to reach a conclusion. In short, this method made it possible to apply algebra and analysis to geometry. Conversely, geometric interpretation offers good insights for solving many algebra and analysis problems. Simultaneous equations may be the simplest example of geometric interpretation. Today the concept of coordinates is taught in junior and senior high school and is very familiar. If you stop to think about it, you will see how important the concept of coordinates has been to the development of mathematics.

1 COORDINATES OF POINTS

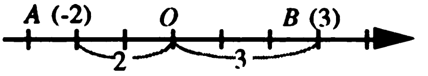
1 Coordinates of Points on a Straight Line

You have already learned in Chapter I that a real number corresponding to a point P on a number line is called the coordinate of P, and that we express the fact that x is the coordinate of P as $P(x)$.

The Distance between Two Points

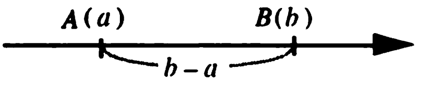
Two points $A(-2)$ and $B(3)$ are given on a number line with origin at O. A is a point to the left of O at a distance of 2 from it. B is a point to the right of O at a distance of 3. Therefore, the distance between A and B is 5. We can also express this distance in terms of coordinates:

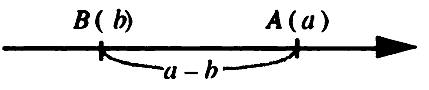
$$AB = 3 - (-2).$$



In general, the distance between two points $A(a)$ and $B(b)$ is expressed using absolute value signs:

$$AB = |b - a|.$$





Problem 1 Three points $A(-4)$, $B(-2)$, and $C(5)$ are given on the number line. Find the distances AO, AB, and BC.

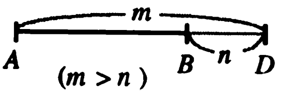
Internal and External Dividing Points

Given point C on line segment AB. If point C satisfies the condition

$$AC : CB = m : n,$$

point C is said to **divide** line segment AB **internally** at a ratio of $m : n$.

Given point D on the extension of line segment AB. If point D satisfies the condition



$(m > n)$

$$AD : DB = m : n,$$

point D is said to **divide** line segment AB **externally** at a ratio of $m : n$.

$(m < n)$

Here we assume that $m > 0$ and $n > 0$ for both internal and external division, and that $m \neq n$ for external division.

Problem 2 Given two points $A(3)$ and $B(7)$. Let P divide line segment AB internally at a ratio of $3 : 1$, let Q divide line segment AB externally at a ratio of $3 : 1$, and let R divide line segment AB externally at a ratio of $1 : 3$. Mark these points on the number line.

Demonstration Given two points $A(a)$ and $B(b)$. Prove that the coordinate c of point C, which divides the line segment AB internally at a ratio of $m : n$, can be expressed as

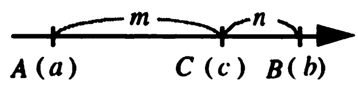
$$c = \frac{mb + na}{m + n}. \tag{1}$$

[Proof]

(i) For $a < b$, since $a < c < b$, we have

$$AC = c - a, \quad CB = b - c.$$

(i)

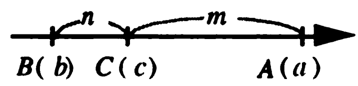


Since $AC : CB = m : n$,

$$(c - a) : (b - c) = m : n$$

$$m(b - c) = n(c - a).$$

(ii)



Therefore, $c = \dfrac{mb + na}{m + n}$.

(ii) For $a > b$ you can construct an analogous proof.

Problem 3 Prove the above Demonstration for $a > b$.

For the special case when point $C(c)$ is the midpoint of line segment AB, we can let $m = n$ in (1), and then we obtain $c = \dfrac{a + b}{2}$.

Problem 4 Given two points $A(-5)$ and $B(7)$. Find the coordinates of the following points:

(1) The midpoint of line segment AB.

(2) An internal dividing point of line segment AB at a ratio of $3 : 2$.

Problem 5 Given two points $A(a)$ and $B(b)$. Prove that the coordinate d of an external dividing point D of line segment AB can be expressed as $d = \dfrac{mb - na}{m - n}$.

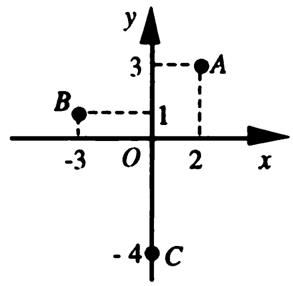
Problem 6 Given two points $A(1)$ and $B(5)$. Find the coordinates of the following points:

(1) A point which divides line segment AB externally at a ratio of $3 : 2$.

(2) A point which divides line segment AB externally at a ratio of $1 : 4$.

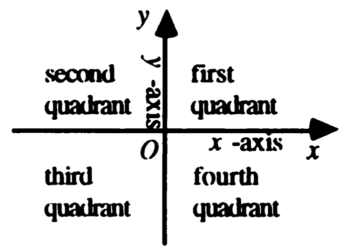
2 Coordinates of Points in a Plane

Every point in a plane can be expressed by coordinates once we define coordinate axes, that is, the x-axis and y-axis in the plane. For example, the coordinates of points A, B, and C at the right are (2, 3), (-3, 1) and (0, -4), respectively.



A plane with coordinate axes is called the **coordinate plane.**

The coordinate plane is divided into four parts. These parts are called the **first quadrant, second quadrant, third quadrant,** and **fourth quadrant,** as in the figure to the right.



Note that the coordinate axes do not belong to any quadrant.

Problem 1 Mark the following points in the coordinate plane. In which quadrant is each point situated?

$$A(-3, 2),\quad B(-5, -1),\quad C(2, -1),\quad D(6, 5)$$

Problem 2 Find the coordinates of the following points, given the point (2, 3):

(1) The point symmetric to it with respect to the x-axis.

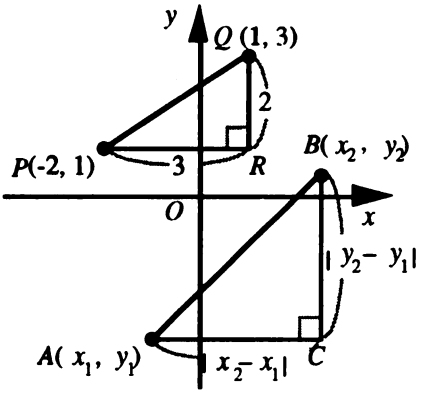
(2) The point symmetric to it with respect to the y-axis.

(3) The point symmetric to it with respect to the origin.

The Distance between Two Points

Let's find the distance between two points $P(-2, 1)$ and $Q(1, 3)$ in the coordinate plane.

First we set up right triangle PQR as in the figure to the right, and we get



$$PR = 3, \ QR = 2.$$

From the Pythagorean theorem,

$$\begin{aligned} PQ &= \sqrt{PR^2 + QR^2} \\ &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13}\,. \end{aligned}$$

Therefore, the distance between the two points is $\sqrt{13}$.

The distance between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ can be found analogously.

Assuming that line segment AB is not parallel to the coordinate axes, we can set up right triangle ABC as above, and then we have

$$AC = |x_2 - x_1|, \ BC = |y_2 - y_1|.$$

Therefore,

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,. \tag{1}$$

Problem 3 Prove that (1) above holds when line segment AB is parallel to the x-axis or to the y-axis.

> **The Distance between Two Points**
>
> The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is
>
> $$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,.$$
>
> As a special case, the distance between the origin and point $P(x, y)$ is
>
> $$OP = \sqrt{x^2 + y^2}$$

Problem 4 Find the distances between the following pairs of points:

(1) (0, 0), (4, -3) (2) (3, -1), (5, 0)

(3) (-3, 2), (-5, -3) (4) (3, -2), (3, -19)

Demonstration 1 If we let M be the midpoint of side BC of ΔABC, then

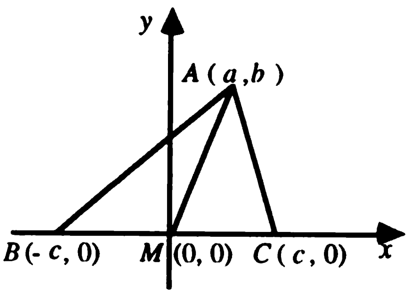
$$AB^2 + AC^2 = 2(AM^2 + BM^2).$$

Prove this.

[Proof] Let us regard line BC as the x-axis and the line through M perpendicular to BC as the y-axis. If we let the coordinates of vertices A, B, and C be

(a, b), $(-c, 0)$, and $(c, 0)$,

respectively, then we have



$$AB^2 + AC^2 = \{(a+c)^2 + b^2\} + \{(a-c)^2 + b^2\}$$
$$= 2(a^2 + b^2 + c^2).$$

Since $AM^2 = a^2 + b^2$, $BM^2 = c^2$,

$$2(AM^2 + BM^2) = 2(a^2 + b^2 + c^2).$$

Therefore,

$$AB^2 + AC^2 = 2(AM^2 + BM^2).$$

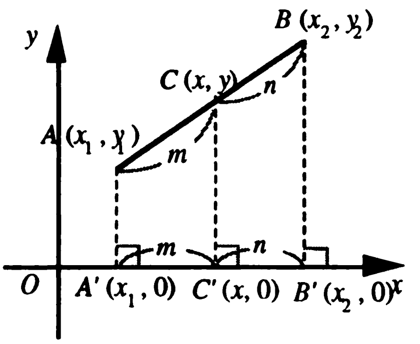
Problem 5 If we let D be a point that divides side BC of ΔABC internally at a ratio of 1 : 2, then

$$2AB^2 + AC^2 = 3(AD^2 + 2BD^2).$$

Prove this.

Internal and External Dividing Points

Let's find the coordinate of point C, which divides line segment AB internally at a ratio of $m : n$ for two points $A(x_1, y_1)$, $B(x_2, y_2)$.



Drawing perpendiculars AA', BB', and CC' to the x-axis from points A, B, and C, we obtain

$$AC : CB = A'C' : C'B' = m : n.$$

Point C' is an internal dividing point of $A'B'$ at a ratio of $m : n$. Since the x-coordinates of points A', B', and C' are x_1, x_2, and x, respectively, the formula for an internal dividing point on the number line is

$$x = \frac{mx_2 + nx_1}{m + n} .$$

Analogously, by drawing a line perpendicular to the y-axis, we obtain

$$y = \frac{my_2 + ny_1}{m + n} .$$

Therefore, the coordinates of point C are

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right).$$

We can determine the coordinates of an external dividing point in an analogous way.

The coordinates of the point that divides line segment AB externally at a ratio of $m : n$ are

$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right).$$

Problem 6 Prove the formula for an external dividing point.

Coordinates of Internal and External Dividing Points

The coordinates of the points that divide line segment AB internally and externally at a ratio of $m : n$ for two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$$

$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right)$$

respectively. As a special case, the coordinates of the midpoint of AB are

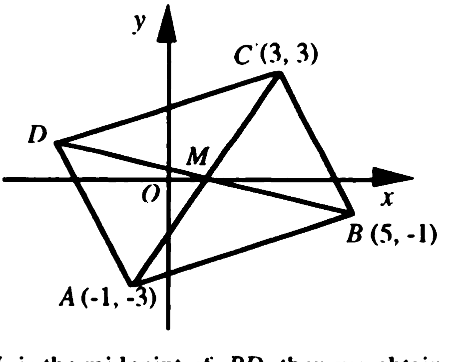
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Problem 7 Find the coordinates of the internal and external dividing points of line segment AB at a ratio of 3 : 2 for the following pairs of points.

(1) A(2, 4), B(5, 1) (2) A(-2, 3), B(4, -1)

Demonstration 2 Find the coordinates of vertex D of a parallelogram $ABCD$ if the coordinates of vertices A, B, and C are (-1, -3), (5, -1), and (3, 3), respectively.

[Solution] We will take advantage of the fact that the diagonals of a parallelogram intersect each other at their midpoints. If we let M be the midpoint of AC, then the coordinates of point M are (1, 0), from the midpoint formula. If we take (x, y) as the coordinates of point D, since point M is the midpoint of BD, then we obtain



$$\frac{5 + x}{2} = 1 , \ \frac{-1 + y}{2} = 0.$$

Therefore,

$$x = -3 , y = 1.$$

Thus,

$$D(-3, 1).$$

Problem 8 Find the coordinates of the point symmetric to the point (1, -1) with respect to the point (2, 3).

Problem 9 Given a triangle whose vertices are the three points $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Prove that the coordinates of the centroid of this triangle are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

Exercises

1. Given two points $A(-2)$ and $B(14)$. Let P be an internal dividing point of AB at a ratio of 5 : 3, and let Q be an external dividing point of line segment AB at a ratio of 7 : 11. Find the coordinates of the midpoint M of line segment PQ.

2. Given two points $A(1)$ and $B(7)$. Let C be an internal dividing point of AB at a ratio of 4 : 3, and let D be an external dividing point of line segment AB at a ratio of 4 : 3. Find the length of CD.

3. Given a triangle whose vertices are $A(6, 5)$, $B(5, 0)$, and $C(-2, 4)$. What kind of triangle is it?

4. Find the coordinates of the three vertices of a triangle in which the midpoints of the sides are (-2, 3), (3, -1), and (5, 4).

5. Find the points on the x- and y-axes which are equidistant from two points $A(-1, 3)$ and $B(2, 4)$.

2 STRAIGHT LINES

1 Equations of Straight Lines

As you have already learned, the equation $y = ax + b$ defines the straight line through the point $(0, b)$ with a slope of a. The constant b is called the **y-intercept**.

The equation $x = k$ defines a straight line through the point $(k, 0)$ parallel to the y-axis.

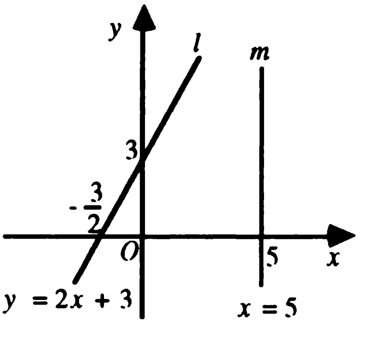
The straight line represented by the equation $y = ax + b$ is the set of all points whose coordinates (x, y), given as pairs of real number values x, y, satisfy this equation. That is,

$$\{(x, y) \mid y = ax + b\}.$$

For example, in the figure to the right,

$$l = \{(x, y) \mid y = 2x + 3\}$$

$$m = \{(x, y) \mid x = 5\}.$$



In general, the pairs of real number values x, y that satisfy an equation in x, y are called the **solutions** of that equation.

The set of all points whose coordinates are solutions of an equation is referred to as the **figure expressed by the equation** or the **graph of the equation**. And that equation is said to be the **equation of the figure**.

Problem 1 Graph the figures expressed by the following equations on the coordinate plane.

(1) $2x + 5y = 0$ (2) $2x - 4y + 5 = 0$

(3) $4x + 12 = 0$ (4) $y - 2 = 0$

Problem 2 Demonstrate that the figure defined by the linear equation

$$ax + by + c = 0$$

is a straight line if either a or b is not equal to 0.

Various Forms for Equations of Straight Lines

Let's find the equation of a straight line through a point (x_1, y_1) with a slope of m.

The equation of a straight line with a slope of m is expressed as

$$y = mx + n. \tag{1}$$

Since this line passes through the point (x_1, y_1)

$$y_1 = mx_1 + n.$$

Therefore,

$$n = y_1 - mx_1.$$

Substituting this value into (1), we obtain

$$y = mx + y_1 - mx_1.$$

That is,

$$y - y_1 = m(x - x_1).$$

The Straight Line through a Point with a Slope of m

The equation of a straight line through a point (x_1, y_1) with a slope of m is

$$y - y_1 = m(x - x_1).$$

Problem 3 Find the equations for straight lines passing through the point (3, 4) and satisfying the following conditions:

(1) With a slope of 2. (2) Parallel to the x-axis.

Next let's find the equation of a straight line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$.

(i) For $x_1 \neq x_2$:

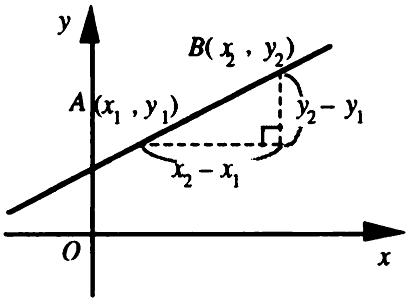
Let m be the slope of line AB. Since this line passes through $A(x_1, y_1)$, the above formula gives us the following equation:

$$y - y_1 = m(x - x_1). \tag{2}$$

However, since this line also passes through the point $B(x_2, y_2)$,

$$y_2 - y_1 = m(x_2 - x_1).$$

Therefore, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.



Substituting this value into (2), we obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

(ii) For $x_1 = x_2$:

Since line AB is parallel to the y-axis, the equation is

$$x = x_1.$$

Straight Lines through Two Points

The equation of a straight line through two points (x_1, y_1) and (x_2, y_2) is

for $x_1 \neq x_2$: $\quad y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$,

for $x_1 = x_2$: $\quad x = x_1$.

Problem 4 Find the equations of straight lines through the following pairs of points:

(1) (2, 3), (10, 5) (2) (5, -1), (-1, 8)

(3) (-2, 0), (-2, 5) (4) (2, 5), (-7, 5)

Problem 5 Prove that the equation of a straight line through two points $(a, 0)$ and $(0, b)$ is $\frac{x}{a} + \frac{y}{b} = 1$. Here we assume $ab \neq 0$.

Problem 6 Find the coordinates of the intersection of two straight lines $x + y - 4 = 0$ and $2x - y + 1 = 0$. Find the equation of the line that passes through this intersection and the point $(-2, 1)$.

2 Parallel and Perpendicular Straight Lines

Let's express the relations between parallel and perpendicular straight lines in terms of the relations among the coefficients of their equations.

Suppose the equations of two straight lines are

$$y = mx + n$$

$$y = m'x + n'.$$

From the meaning of the slope, if these are parallel, then

$$m = m'.$$

Conversely, if $m = m'$, then these two lines are parallel.

If both $m = m'$ and $n = n'$, these two lines are coincident. Let us regard this as a special case of parallel lines.

Problem 1 Which of the following lines are parallel?

(1) $6x + 3y = 1$ (2) $y = 3x - 2$

(3) $2x - y = 5$ (4) $y = -2x + 5$

(5) $2x = y - 2$

Next, for $m \neq 0$ and $m' \neq 0$, let's consider what happens if the following two lines are perpendicular:

$$y = mx + n$$

$$y = m'x + n'.$$

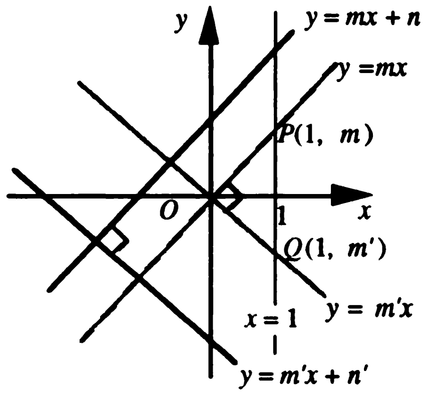
We can deal with this problem by translating the two lines so that they pass through the origin

$$y = mx \text{ and } y = m'x$$

and finding the relation between the coefficients that indicates perpendicularity.

Let P and Q be the points at which these lines intersect the line $x = 1$. If $OP \perp OQ$, then from the Pythagorean theorem,

$$PQ^2 = OP^2 + OQ^2.$$



Since the coordinates of P and Q are $(1, m)$ and $(1, m')$, respectively, the above equation can be rewritten as

$$(m - m')^2 = (1 + m^2) + (1 + m'^2)$$

Rearranging this equation, we obtain

$$mm' = -1.$$

Conversely,

if $mm' = -1$, then $OP \perp OQ$.

Conditions for Parallel and Perpendicular Straight Lines

Given two straight lines $y = mx + n$ and $y = m'x + n'$:

they are parallel if $m = m'$

and they are perpendicular if $mm' = -1$.

Demonstration 1 Find the equations of straight lines through the point (-1, 4) that are parallel and perpendicular to the line $2x - 3y = 6$.

[Solution] The equation of the given line can be transformed into

$$y = \frac{2}{3}x - 2.$$

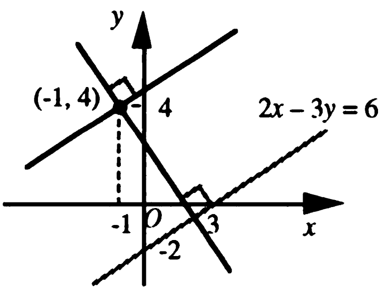
The slope of a straight line parallel to this line is $\frac{2}{3}$, and the slope of a perpendicular line is $-\frac{3}{2}$.

Therefore, the equation of a line through point (-1, 4) parallel to the given line is

$$y - 4 = \frac{2}{3}(x + 1).$$

Thus,

$$2x - 3y + 14 = 0.$$



The equation of a perpendicular line is

$$y - 4 = -\frac{3}{2}(x + 1).$$

Thus,

$$3x + 2y - 5 = 0.$$

Problem 2 Find the equations of the following lines:

(1) A straight line through the point (-1, 2) parallel to $3x + 2y = 5$.

(2) A straight line through the point (1, 1) perpendicular to a line through the two points (3, 4) and (-2, 7).

Problem 3 Given the two straight lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$, prove that:

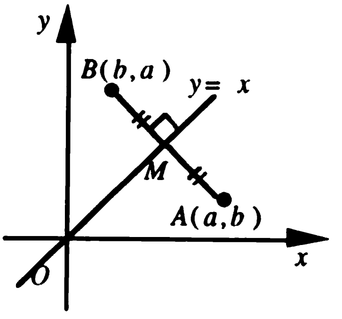
(1) They are parallel if $ab' - a'b = 0$.

(2) They are perpendicular if $aa' + bb' = 0$.

Demonstration 2 Prove that the two points $A(a, b)$ and $B(b, a)$ are symmetric with respect to the line $y = x$. Here we assume that $a \neq b$.

[Proof] The slope m of line $y = x$ is

$$m = 1.$$



The slope m' of the straight line through the two points A and B is

$$m' = \frac{a-b}{b-a} = -1.$$

Therefore,

$$mm' = -1.$$

Thus, line AB and the line $y = x$ are perpendicular to each other.

Now suppose M is the midpoint of line AB. Since the coordinates of M are

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$$

M must lie on the line $y = x$.

Thus, point $A(a, b)$ and point $B(b, a)$ are symmetric with respect to the line $y = x$.

Problem 4 Find the coordinates of point P, which is symmetric to the origin O with respect to line $x - 3y + 6 = 0$, using conditions (i) and (ii).

(i) Line OP is perpendicular to the line $x - 3y + 6 = 0$.

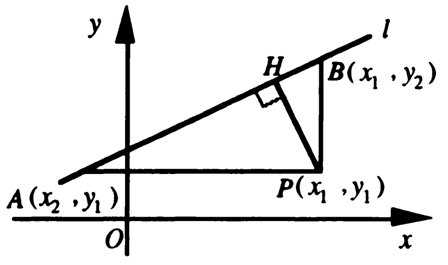
(ii) The midpoint of line segment OP lies on the line $x - 3y + 6 = 0$

The Distance between a Point and a Straight Line

Let l be a straight line $ax + by + c = 0$, and let $P(x_1, y_1)$ be a point which is not located on l. Let's prove that the distance between point P and line l, that is, the length of the perpendicular PH from point P to line l, is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

Here let us assume that $a \neq 0$ and $b \neq 0$. Let $A(x_2, y_1)$ and $B(x_1, y_2)$ be the points at which line l intersects lines through P parallel to the x-axis and y-axis, respectively. And let



$$ax_1 + by_1 + c = k. \qquad (1)$$

Since points A and B lie on line l,

$$ax_2 + by_1 + c = 0 \qquad (2)$$

$$ax_1 + by_2 + c = 0. \qquad (3)$$

Subtracting (2) and (3) from (1), respectively, we obtain

$$a(x_1 - x_2) = k, \quad b(y_1 - y_2) = k.$$

Therefore,

$$x_1 - x_2 = \frac{k}{a}, \quad y_1 - y_2 = \frac{k}{b}. \qquad (4)$$

Since $PH \cdot AB$ and $PA \cdot PB$ are both equal to twice the area of ΔPAB,

$$PH \cdot AB = PA \cdot PB. \qquad (5)$$

From (4) and (5),

$$PH^2 = \frac{PA^2 \cdot PB^2}{AB^2} = \frac{(x_1 - x_2)^2 \cdot (y_1 - y_2)^2}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \frac{\frac{k^2}{a^2} \cdot \frac{k^2}{b^2}}{\frac{k^2}{a^2} + \frac{k^2}{b^2}} = \frac{k^2}{a^2 + b^2}.$$

Therefore,

$$PH = \frac{|k|}{\sqrt{a^2 + b^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

The Distance between a Point and a Straight Line

The distance d between a point (x_1, y_1) and a straight line $ax + by + c = 0$ is

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

This formula also holds for $a = 0$ or $b = 0$.

Problem 5 Find the distance between the following points and lines using the above formula.

(1) The point $(3, 4)$ and the line $x + 2y - 2 = 0$.

(2) The origin and the line $4x - 3y = 5$.

(3) The point $(-1, 3)$ and the line $y = \frac{1}{4}x + 2$.

Problem 6 Find the answer to Problem 5, part (1) by the following method.

(i) Let l be the straight line $x + 2y - 2 = 0$. Let $H(\alpha, \beta)$ be the point at which line l intersects a perpendicular line through $P(3, 4)$ to l. Make up simultaneous equations involving α and β, utilizing the fact that PH is perpendicular to l and H lies on l.

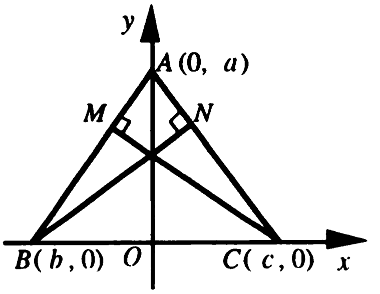
(ii) Find the value of α and β by solving the simultaneous equations, then calculate the length of PH.

Using Coordinates to Prove Properties of Figures

Demonstration 3 Prove that the three perpendiculars from the vertices of ΔABC to the opposite sides intersect at a single point.

[Proof] Let line BC be the x-axis, let the perpendicular line AO from A to BC be the y-axis, and let the coordinates of the vertices be

$A(0, a)$, $B(b, 0)$, $C(c, 0)$.



(i) For $b \neq 0$ and $c \neq 0$:

Since the slope of line AB is $-\frac{a}{b}$, the equation of the perpendicular line CM from C to side AB is

$$y = \frac{b}{a}(x - c) = \frac{b}{a}x - \frac{bc}{a}.$$

Since the slope of line AC is $-\frac{a}{c}$, the equation of the perpendicular line BN from B to side AC is

$$y = \frac{c}{a}(x - b) = \frac{c}{a}x - \frac{bc}{a}.$$

Since the y-intercepts of both lines CM and BN are $-\frac{bc}{a}$, CM and BN intersect at a point $(0, -\frac{bc}{a})$ on the y-axis.

In other words, they intersect at a single point on perpendicular line AO.

(ii) For $b = 0$ or $c = 0$:

Since ΔABC is a right triangle, it is clear that the three perpendicular lines intersect at the vertex that forms the right angle.

The intersection in Demonstration 3 is called the **orthocenter** of ΔABC.

When you use coordinates to prove properties of figures, as in Demonstration 3, you can conduct the proof by the following procedure.

(a) Define proper coordinate axes.

(b) Represent the figure and its relation in terms of a numerical expression involving coordinates.

(c) Perform computations on the numerical expression derived in (b).

Problem 7 Prove that the perpendicular bisectors of the three sides of ΔABC intersect at a single point.

The intersection of the perpendicular bisectors in Problem 7 is called the **circumcenter** of ΔABC.

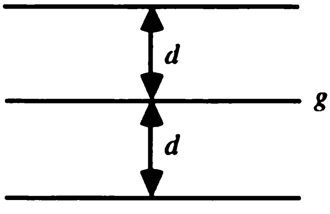
1 Equation of a Locus

When a point moves in accordance with a certain condition, the figure described by that point is called the **locus** of the points satisfying the condition. In other words, a locus is a set of all points which satisfy a given condition.

Example 1 The locus of the points at a constant distance d from a fixed line g, that is,

$\{P \mid$ (the distance between point P and line $g) = d\}$

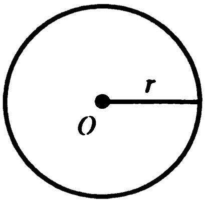
is two straight lines parallel to line g.



Example 2 The locus of the points at a constant distance r from a fixed point C, that is,

$$\{P \mid OP = r\}$$

is a circle of radius r with center O.



Problem 1 What is the locus of points which satisfy the following conditions:

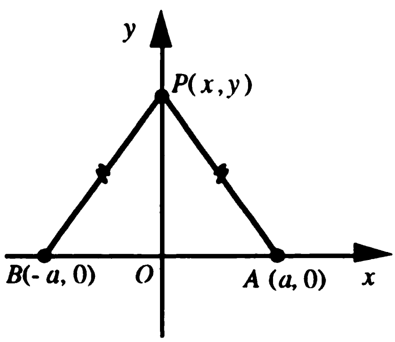
(1) The points equidistant from two intersecting lines.

(2) The point P such that $\angle APB$ is a right angle for two fixed points A and B.

In order to find a locus of points using coordinates, we should first express the given condition in terms of coordinates and then derive an equation for the locus.

Demonstration 1 Find the locus of points P equidistant from two fixed points A and B.

[Solution] Take line AB as the x-axis, and take the perpendicular bisector of AB as the y-axis. Let $(a, 0)$ and $(-a, 0)$ be the coordinates of fixed points A and B, respectively, and take (x, y) as the coordinates of moving point P.



(1) The given condition is

$$PA = PB.$$

That is,

$$PA^2 = PB^2.$$

The distance formula gives us

$$(x-a)^2 + y^2 = (x+a)^2 + y^2.$$

Rearranging this equation, we obtain

$$4ax = 0.$$

Since $a \neq 0$,

$$x = 0.$$

Therefore, point P lies on line $x = 0$, the y-axis.

(2) Conversely, if we take $Q(0, y)$ as any point on the y-axis, then

$$QA = \sqrt{a^2 + y^2}, \quad QB = \sqrt{a^2 + y^2}.$$

Therefore, $$QA = QB$$

In other words, any point on the y-axis is equidistant from two fixed points A and B.

From (1) and (2) we know that the locus of point P is a perpendicular bisector of line segment AB.

In general, to demonstrate that a figure F is a locus of points that satisfy a given condition, it is sufficient to prove the following two assertions:

(1) Any point satisfying the given condition lies on figure F.

(2) All the points on figure F satisfy the given condition.

The preceding Demonstration was solved by this method. However, when we use coordinates, we can often see that the converse holds simply by reversing the procedure used to prove (1). Hence, the proof of (2) is usually omitted.

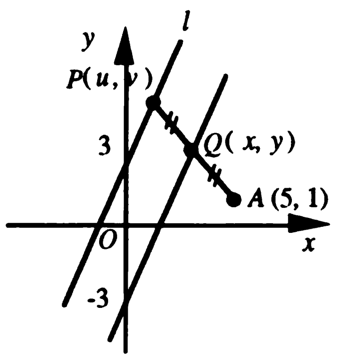
Problem 2 Given two fixed points $A(2, 0)$ and $B(-2, 0)$. Find the locus of the points satisfying the equation

$$AP^2 - BP^2 = 16.$$

Demonstration 2 Let l be the line $y = 2x + 3$, and let $A(5, 1)$ be a fixed point which does not lie on l. Find the locus of the midpoint Q of line segment AP when point P moves along l.

[Solution] Let (u, v) be the coordinates of point P. Since this point lies on l,

$$v = 2u + 3. \qquad (1)$$



On the other hand, let (x, y) be the coordinates of point Q. Since Q is the midpoint of AP,

$$x = \frac{u + 5}{2}, \quad y = \frac{v + 1}{2}.$$

Thus,

$$u = 2x - 5, \quad v = 2y - 1.$$

Substituting these equalities into (1), we obtain

$$2y - 1 = 2(2x - 5) + 3.$$

Rearranging this equation, we get

$$y = 2x - 3.$$

Hence, the locus of point Q is a straight line $y = 2x - 3$.

Problem 3 Given point $A(-3, 1)$, let P move along the straight line $2x - y - 1 = 0$. Find the locus of point Q, which divides line segment AP internally at a ratio of $3 : 5$.

Exercises

1. Find the equations of the following straight lines:

   (1) A straight line through the point (2, 5) with a slope of -3.

   (2) A straight line through the two points (-5, 3) and (2, -1).

   (3) A straight line through the point (3, -4) parallel to the line $2x - y = 5$.

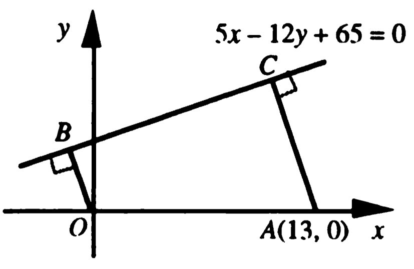
   (4) A straight line through the point (-1, 1) perpendicular to the line $x + 3y + 4 = 0$.

2. Find the equation of a line that passes through the origin and the intersection of the two lines $2x - y - 1 = 0$ and $3x + 2y - 2 = 0$.

3. Find the value of k such that two lines $3x + 4y - 5 = 0$ and $4x + ky - 4 = 0$ intersect perpendicularly.

4. Find the equation of the line which is symmetric to the line $y = 3x + 2$ with respect to the line $y = x$.

5. Let OB and AC be perpendicular lines from the origin and point $A(13, 0)$ to line $5x - 12y + 65 = 0$. Find the area of trapezoid $OACB$.



6. Given three points $O(0, 0)$, $A(2, 1)$, and $B(3, 2)$. Find the locus of point P which satisfies

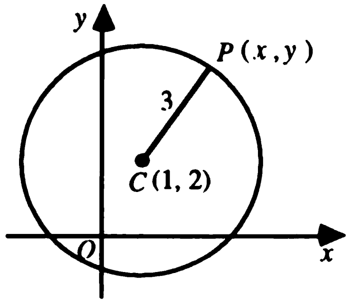
$$PO^2 + PA^2 = 2PB^2.$$

3 CIRCLES

1 Equations of Circles

Let's find the equation of a circle with its center at point $C(1, 2)$ and a radius of 3. Since this circle is the locus of points at a distance of 3 from point C, if we let $P(x, y)$ be a point on the locus, then the condition of the locus, $CP = 3$, can be rewritten as

$$\sqrt{(x-1)^2 + (y-2)^2} = 3.$$



In other words, the condition can be expressed as

$$(x-1)^2 + (y-2)^2 = 9.$$

This is an equation for a circle with a radius of 3 and center at (1, 2). The general form of the equation of a circle is given below.

> **The Equation of a Circle**
>
> The equation of a circle with a radius of r and center at a point (a, b) is
>
> $$(x-a)^2 + (y-b)^2 = r^2.$$
>
> The special case of a circle with a radius of r and center at the origin has the equation
>
> $$x^2 + y^2 = r^2.$$

Note: A circle whose equation is $(x-a)^2 + (y-b)^2 = r^2$ is sometimes designated simply as

the circle $(x-a)^2 + (y-b)^2 = r^2$.

Problem 1 Find the equations of the following circles:

(1) A circle with a radius of $\sqrt{3}$ and center at (-2, 1).

(2) A circle with center at (3, -2) which passes through the origin.

(3) A circle through the two points (7, 3) and (-1, -3) as the endpoints of a diameter.

(4) A circle through the two points (3, $\sqrt{3}$) and (2, -2) with its center on the x-axis.

Demonstration 1 What figure does the following equation define?

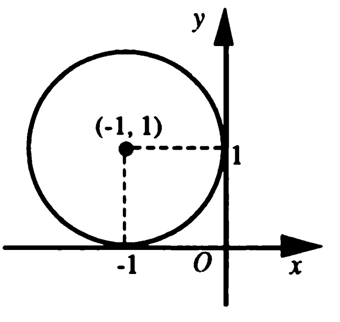
$$x^2 + y^2 + 2x - 2y + 1 = 0$$

[Solution] Transforming the given equation, we obtain

$$(x^2 + 2x) + (y^2 - 2y) = -1$$

$$(x + 1)^2 + (y - 1)^2 = 1.$$

This defines the circle with a radius of 1 and center at the point (-1, 1).



In general, circles are defined by quadratic equations in x and y of the following form where the coefficients of x^2 and y^2 are identical and there is no term in xy:

$$x^2 + y^2 + Ax + By + C = 0.$$

Problem 2 What figures do the following equations represent?

(1) $x^2 + y^2 - 6x + 4y + 4 = 0$

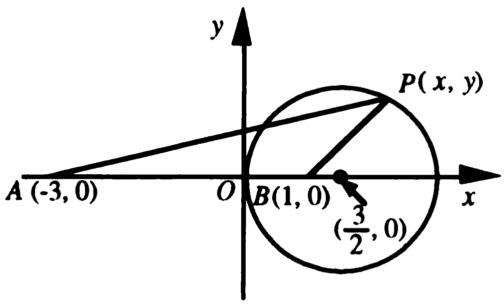
(2) $3x^2 + 3y^2 - 2x - 3y + 1 = 0$

Problem 3 Determine the value of the coefficients A, B, and C such that the circle $x^2 + y^2 + Ax + By + C = 0$ passes through the three points (-6, 6), (-2, 8), and (1, -1).

Demonstration 2 Given two fixed points $A(-3, 0)$ and $B(1, 0)$. Find the locus of the points which satisfy

$$AP : PB = 3 : 1.$$

[Solution] Let (x, y) be the coordinates of point P. Since the condition satisfied by point P is



$AP : PB = 3 : 1,$

we have

$AP = 3PB.$

Squaring both sides, we get

$$AP^2 = 9PB^2.$$

From the formula for the distance between two points,

$$(x + 3)^2 + y^2 = 9\{(x - 1)^2 + y^2\}.$$

Rearranging this equation,

$$x^2 - 3x + y^2 = 0$$

$$(x - \frac{3}{2})^2 + y^2 = (\frac{3}{2})^2.$$

Therefore, the locus of point P is a circle with a radius of $\frac{3}{2}$ and center at $(\frac{3}{2}, 0)$.

Note: In general, given two fixed points A and B, the locus of point P which satisfies $AP : PB = m : n$ for $m \neq n$ is a circle. This circle is called "Apollonius's circle".

Problem 4 Given two fixed points $A(-2, 0)$ and $B(3, 0)$. Find the locus of point P satisfying $AP : PB = 3 : 2$.

Problem 5 Given three fixed points $A(1, 0)$, $B(-2, 3)$, and $C(-5, -3)$. Find the locus of point P satisfying $AP^2 = BP^2 + CP^2$.

Problem 6 Find the locus of the point such that the sum of the squares of its distance from two fixed points A and B is a constant value $2k^2$ for $AB < 2k$.

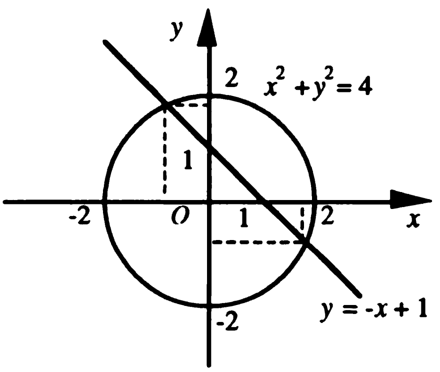
1 Circles and Straight Lines

Common Points of Circles and Straight Lines

The coordinates of the common points of two straight lines are given by solving simultaneous equations expressing these two lines. Now let's examine relations between circles and lines in a similar way using equations.

Example The common points of the straight line $y = -x + 1$ and the circle $x^2 + y^2 = 4$ can be found by solving these simultaneous equations:

$$\begin{cases} y = -x + 1 \\ x^2 + y^2 = 4 \end{cases}.$$



Solving them, we find that the coordinates of the common points are

$$\left(\frac{1+\sqrt{7}}{2}, \frac{1-\sqrt{7}}{2}\right)$$

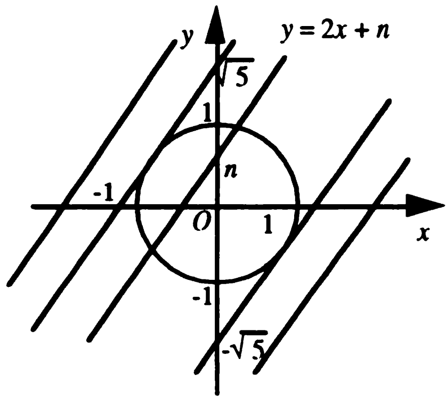
$$\left(\frac{1-\sqrt{7}}{2}, \frac{1+\sqrt{7}}{2}\right).$$

Problem 1 Find the coordinates of the points shared by the line $2x - y - 2 = 0$ and the circle $x^2 + y^2 - x + 4y = 0$.

Demonstration 1 How does the number of points shared by the circle $x^2 + y^2 = 1$ and the line $y = 2x + n$ change as the value of n changes?

[Solution] The number of common points is the same as the number of real solutions to the simultaneous equations:

$$\begin{cases} x^2 + y^2 = 1 & (1) \\ y = 2x + n & (2) \end{cases}$$



Substituting (2) into (1) and rearranging it, we obtain

$5x^2 + 4nx + n^2 - 1 = 0.$ (3)

Let D be the discriminant of this equation.

$$\frac{D}{4} = 4n^2 - 5(n^2 - 1) = 5 - n^2$$

Therefore, equation (3) has:

for $D > 0$ or $-\sqrt{5} < n < \sqrt{5}$: two real solutions;

for $D = 0$ or $n = \pm\sqrt{5}$: a multiple solution;

for $D < 0$ or $n < -\sqrt{5}$, $\sqrt{5} < n$: imaginary solutions.

Thus,

for $-\sqrt{5} < n < \sqrt{5}$: there are two common points;

for $n = \pm\sqrt{5}$: there is one common point;

for $n < -\sqrt{5}$, $\sqrt{5} < n$: there are no common points.

In this Demonstration, for the special case of $n = \pm\sqrt{5}$ the line is tangent to the circle.

Problem 2 Find the value of n if the line $y = x + n$ is tangent to the circle $x^2 + y^2 = 4$.

Problem 3 Find the range of the values of m if the line $y = mx - 2$ and the circle $x^2 + y^2 = 1$ intersect at two different points. Find the value of m if they are tangent to each other.

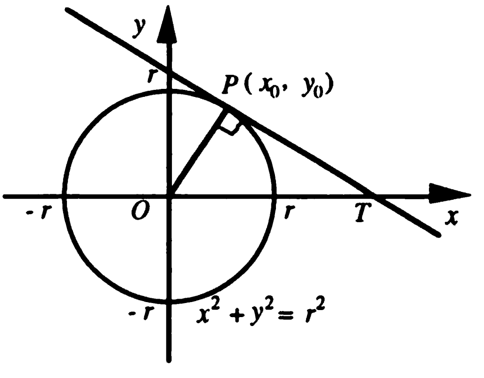
Tangents of Circles

A tangent of a circle is perpendicular to the radius at the tangent point. We can find the equation of a tangent by exploiting this property.

Take point $P(x_0, y_0)$ on the circumference of a circle $x^2 + y^2 = r^2$.

(i) For $x_0y_0 \neq 0$:

Since the slope of the radius OP is $\frac{y_0}{x_0}$, the slope of the tangent at point P is $-\frac{x_0}{y_0}$. Therefore, the equation of tangent PT is



$$y - y_0 = -\frac{x_0}{y_0}(x - x_0).$$

That is,

$$x_0x + y_0y = x_0^2 + y_0^2.$$

Since point $P(x_0, y_0)$ lies on the circumference,

$$x_0^2 + y_0^2 = r^2.$$

Thus, the equation of the tangent is

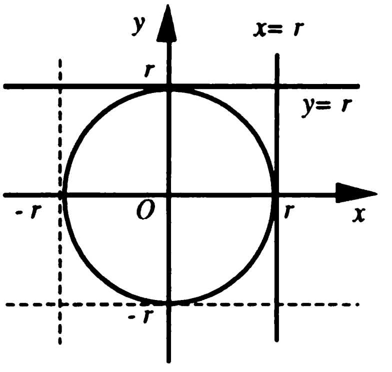
$$x_0x + y_0y = r^2.$$

(ii) For $x_0 y_0 = 0$:

Point P lies on the coordinate axes. For example, for $P(r, 0)$ the equation of the tangent is clearly $x = r$. This is identical to the equation above for

$$x_0 = r, \quad y_0 = 0.$$

We can derive equations for $P(0, r)$, etc., analogously.



We obtain the following generalization:

> **Tangent of a Circle**
>
> The equation of a tangent to a circle $x^2 + y^2 = r^2$ at a point (x_0, y_0) on the circumference is
>
> $$x_0 x + y_0 y = r^2.$$

Problem 4 Find the equations of the tangents to the circle $x^2 + y^2 = 25$ at the points (3, 4), (0, 5), and (-4, 3) on the circumference.

Demonstration 2 Find the equation of a straight line of slope 2 which is tangent to the circle $x^2 + y^2 = 4$.

[Solution] If we let (x_0, y_0) be the tangent point, the equation of the tangent is

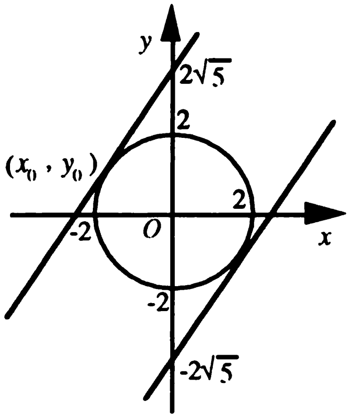
$$x_0 x + y_0 y = 4. \qquad (1)$$

Since the slope is 2,

$$-\frac{x_0}{y_0} = 2. \qquad (2)$$

However,

$$x_0^2 + y_0^2 = 4. \qquad (3)$$



Solving (2) and (3), we obtain

$$\begin{cases} x_0 = \dfrac{4}{\sqrt{5}} \\ y_0 = -\dfrac{2}{\sqrt{5}} \end{cases} \qquad \begin{cases} x_0 = -\dfrac{4}{\sqrt{5}} \\ y_0 = \dfrac{2}{\sqrt{5}} \end{cases}.$$

Substituting these expressions into (1) and rearranging the equation, we get

$$y = 2x - 2\sqrt{5}, \quad y = 2x + 2\sqrt{5}.$$

These are the equations of the tangents.

[Alternate Solution]

A line with a slope 2 is defined by the equation

$$y = 2x + n.$$

Substituting this equation into $x^2 + y^2 = 4$, we get

$$5x^2 + 4nx + n^2 - 4 = 0.$$

The line is tangent to the circle if this equation has a multiple solution:

$$4n^2 - 5(n^2 - 4) = 0.$$

Solving this equation for n, we obtain

$$n = \pm 2\sqrt{5}.$$

Therefore, the equation of the tangents is

$$y = 2x \pm 2\sqrt{5}.$$

Problem 5 The equation of the tangents of a circle $x^2 + y^2 = r^2$ of slope m is

$$y = mx \pm r\sqrt{1 + m^2}.$$

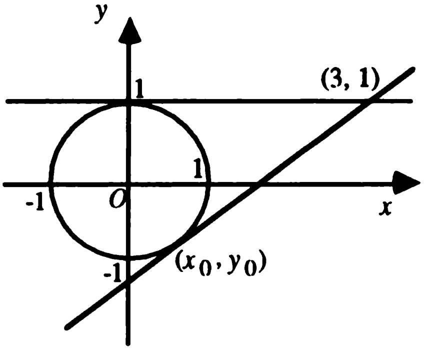
Prove this formula by the method followed in the Alternate Solution above.

Problem 6 Find the equation of a line tangent to the circle $x^2 + y^2 = 9$ and perpendicular to the line $3x + y = 3$.

Demonstration 3 Find the equation of the tangent to circle $x^2 + y^2 = 1$ through the point (3, 1).

[Solution] If we let (x_0, y_0) be a tangent point, then the equation of the tangent line is

$$x_0x + y_0y = 1. \qquad (1)$$



Since this line passes through the point (3, 1),

$$3x_0 + y_0 = 1. \qquad (2)$$

Since (x_0, y_0) is a point on the circle $x^2 + y^2 = 1$,

$$x_0^2 + y_0^2 = 1. \qquad (3)$$

Solving (2) and (3),

$$\begin{cases} x_0 = 0 \\ y_0 = 1 \end{cases} \qquad \begin{cases} x_0 = \frac{3}{5} \\ y_0 = -\frac{4}{5} \end{cases}.$$

Substituting these equalities into (1) and rearranging the equation, we obtain

$$y = 1, \quad 3x - 4y = 5.$$

These are the two equations of the tangents.

Problem 7 Find the equations of the lines tangent to the circle $x^2 + y^2 = 4$ through the point (5, 0).

3 Translation

Let's start out by considering the translation of circles and straight lines, and then move on to consider the translation of figures in general.

Translation of Circles

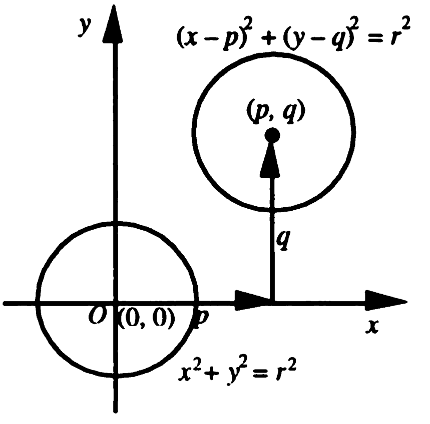
The equation of a circle with a radius of r and its center at the origin is

$$x^2 + y^2 = r^2. \qquad (1)$$

If we translate this circle p units along the x-axis and q units along the y-axis, then it becomes a circle of radius r and center (p, q). The new equation is

$$(x - p)^2 + (y - q)^2 = r^2. \qquad (2)$$

(2) is derived by substituting $x - p$ for x and $y - q$ for y in (1).



Note: Translation of a figure 2 units along the x-axis means translating it 2 units in the positive direction, while a translation of -2 units along the x-axis means translating the figure 2 units in the negative direction.

Problem 1 Given the following pairs of circles. State what translation turns (a) into (b).

(1) (a) $x^2 + y^2 = 1$ (b) $(x - 1)^2 + (y - 2)^2 = 1$

(2) (a) $(x + 2)^2 + (y - 3)^2 = 5$ (b) $(x - 1)^2 + y^2 = 5$

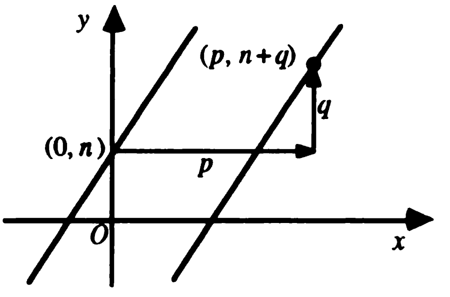
Problem 2 Find the equation of a circle created by translating the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ by 2 units along the x-axis and -1 units along the y-axis.

Translation of Straight Lines

The equation of a straight line of slope m passing through the point $(0, n)$ is

$$y = mx + n. \tag{3}$$

If we translate this line p units along the x-axis and q units along the y-axis, then it becomes a line of slope m passing through the point $(p, n + q)$. The equation of this new line is



$$y - (n + q) = m(x - p).$$

That is,

$$y - q = m(x - p) + n. \tag{4}$$

Equation (4) is derived by substituting $x - p$ for x and $y - q$ for y in (3).

Problem 3 Find the equation of a straight line created by translating the line $y = 2x + 3$ by 4 units along the x-axis and -3 units along the y-axis.

Problem 4 We want to translate the line $y = 3x - 1$ by a units along the x axis and $2a$ units along the y-axis. Find the value of a, if this new line passes through the origin.

Translation of Figures

A circle is defined by an equation of the form $x^2 + y^2 + Ax + By + c = 0$, and a straight line is defined by an equation of the form $ax + by + c = 0$. Both are given in terms of $f(x, y)$, an expression in x and y:

$$f(x, y) = 0.$$

Let's consider the general translation of a figure F defined by an equation $f(x, y) = 0$.

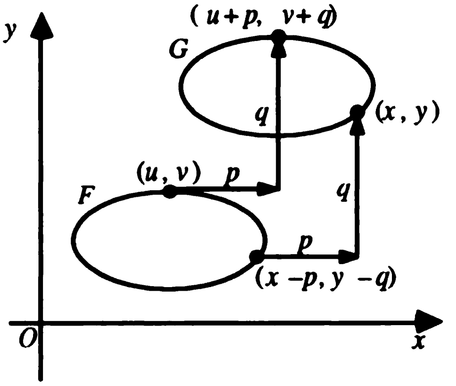
Let G be a figure created by translating figure F by p units along the x-axis and q units along the y-axis. What is the equation of figure G?

This translation causes a point (u, v) on F to move to the point $(u+p, v+q)$ on G.

Therefore, the point $(x-p, y-q)$ on F moves to the point (x, y) on G. Since the point $(x-p, y-q)$ lies on F, we have

$$f(x-p, y-q) = 0.$$

This is the equation satisfied by the point (x, y) on G. In other words, it is the equation of figure G.



Translation of Figures

Figure G is created by translating figure F, defined by the equation $f(x, y) = 0$, p units along the x-axis and q units along the y-axis. The equation of figure G is given by

$$f(x-p, y-q) = 0.$$

Problem 5 Find the equation of the line created by translating the line $x = c$ by p units along the x-axis and q units along the y-axis.

Problem 6 Find the equation of the line created by translating the line $ax + by + c = 0$ by p units along the x-axis and q units along the y-axis.

Problem 7 Find the equation of the figures created by translating the figures defined by the following equations 2 units along the x-axis and -1 unit along the y-axis.

(1) $x + 2y - 3 = 0$

(2) $x^2 + y^2 = 1$

(3) $x^2 - y = 0$

Exercises

1. Find the equations of the following circles:

   (1) A circle with center at (2, -1) and passing through the point (-1, 3).

   (2) A circle passing through the two points (-4, 1) and (4, 5) whose center lies on the straight line $2x - y = 1$.

   (3) A circle passing through the point (1, 2) and tangent to both coordinate axes.

2. Given three points $A(0, 0)$, $B(4, 2)$, and $C(5, 1)$. Find the locus of points satisfying

$$AP^2 + BP^2 + CP^2 = 49.$$

3. Check that the point (5, 4) lies on the circle $(x-2)^2 + (y-3)^2 = 10$. Find the equation of a tangent line at this point by utilizing the concept of translation.

4. Find the equation of the tangents to the circle $x^2 + y^2 = 25$ from the point (1, 7) outside the circle.

5. Find the length of the chord cut from the line $y = x - 1$ by the circle $x^2 + y^2 = 4$.

6. How does the number of points shared by the line $y = mx + 2$ and the circle $x^2 + y^2 = 1$ vary according to the value of m?

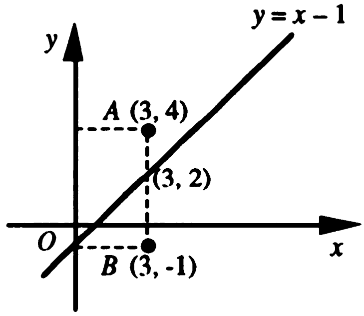
7. Find the equation of a circle passing through the three points (0, 0), (1, 2), and (4, 3).

8. Let P be a moving point on a circle passing through the origin and with its center at (1, 0). Find the locus of the midpoint Q of line segment OP connecting the origin O and point P.

4 REGIONS REPRESENTED BY INEQUALITIES

1 Regions Represented by Inequalities

The straight line $y = x - 1$ is the set of all points (x, y) which satisfy this equation. What other relation is expressed by the coordinates of these points besides a line?



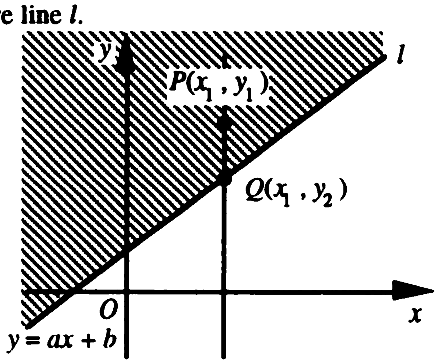
Let's consider the point $A(3, 4)$. Since $x = 3$ and $y = 4$, $y > x - 1$. At point $B(3, -1)$, $y < x - 1$. Let's consider the general implications of these facts.

Let l be a straight line $y = ax + b$. Let $P(x_1, y_1)$ be a point not on line l, and let $Q(x_1, y_2)$ be the point at which line l intersects a line through P parallel to the y-axis.

If point P lies above line l, then

$$y_1 > y_2. \tag{1}$$

Conversely, if (1) holds, then point P lies above line l.



Since point $Q(x_1, y_2)$ lies on the line $y = ax + b$,

$$y_2 = ax_1 + b.$$

Therefore, the necessary and sufficient condition for point $P(x_1, y_1)$ to lie above line l is

$$y_1 > ax_1 + b.$$

In other words, the set of all points (x, y) that satisfy the linear inequality

$$y > ax + b.$$

is the set of all points above the line $y = ax + b$.

Analogously, the set of all points (x, y) that satisfy the linear inequality

$$y < ax + b.$$

is the set of all points below the line $y = ax + b$.

Thus, given an inequality in x and y, the set of all points (x, y) which satisfy the inequality is called **the region represented by the inequality**.

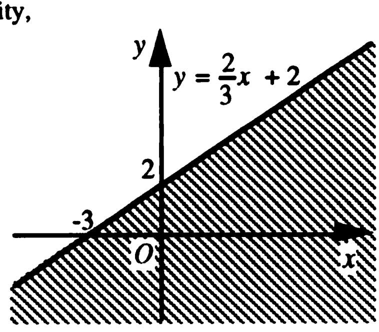
Demonstration 1 Sketch the region represented by the inequality $2x - 3y + 6 > 0$.

[Solution] Transforming the given inequality,

$$y < \frac{2}{3}x + 2.$$

Therefore, the region is the area below the line

$$y = \frac{2}{3}x + 2.$$



Thus, the region is the shaded area in the figure to the right.
The line at the border of the region is not included.

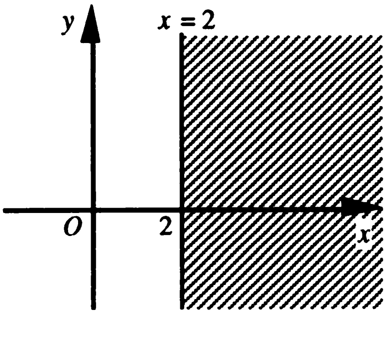
Problem 1 Sketch the regions represented by the following inequalities:

(1) $y > 3x - 2$ (2) $2x + 4y \leq 5$

Demonstration 2 Sketch the region represented by the inequality $x > 2$.

[Solution] The region is the set of all points (x, y) whose x coordinates satisfy $x > 2$ and whose y coordinates are any numbers. Therefore, the region is the area to the right of the straight line

$$x = 2.$$



That is, the region is the shaded area in the figure. Again, the border line is not included.

Problem 2 Sketch the regions represented by the following inequalities:

(1) $x < -3$ (2) $y \geq 2$ (3) $2x - 5 > 0$

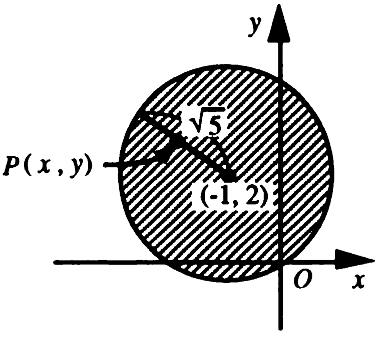
Demonstration 3 Sketch the region represented by the inequality $x^2 + y^2 + 2x - 4y \leq 0$.

[Solution] Transforming the created inequality,

$$(x + 1)^2 + (y - 2)^2 \leq 5.$$

This inequality is satisfied by all points at a distance of no more than $\sqrt{5}$ from point (-1, 2), the center of the circle

$$(x + 1)^2 + (y - 2)^2 = 5.$$



So $\sqrt{5}$ is the radius of this circle. Thus, the points on the circumference and inside the circle satisfy the inequality.

Therefore, the region is the shaded area in the figure above. In this case the border is included.

We can formulate the following generalization analogously.

Inequalities and the Areas inside and outside a Circle

The region represented by $(x - a)^2 + (y - b)^2 < r^2$ is the area inside a circle with center (a, b) and radius r.

The region represented by $(x - a)^2 + (y - b)^2 > r^2$ is the area outside a circle with center (a, b) and radius r.

Problem 3 Sketch the region represented by the following inequalities:

(1) $x^2 + y^2 \geq 4$ (2) $x^2 + y^2 + 4x + 2y + 1 < 0$

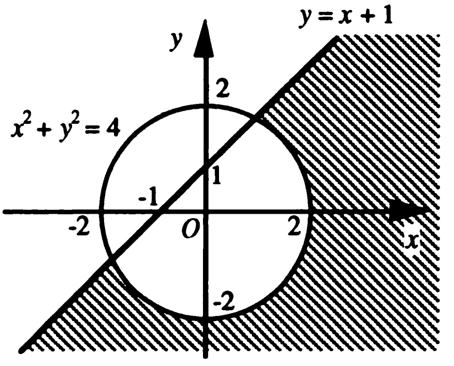
2 Regions Represented by Simultaneous Inequalities

The set of points which simultaneously satisfy two inequalities is the common area of the regions represented by the two inequalities. Let's consider the set of points that simultaneously satisfy the following inequalities:

$$x^2 + y^2 > 4 \tag{1}$$

$$y < x + 1. \tag{2}$$

Let M and N be the regions represented by (1) and (2), respectively. M is the area outside the circle $x^2 + y^2 = 4$. N is the area below the line $y = x + 1$. Since any point that simultaneously satisfies (1) and (2) belongs to both M and N, the set of such points is the common area $M \cap N$ of M and N.



When we sketch the set $M \cap N$ in a diagram, we get the shaded area in the figure to the right. The border is not included. This figure represents the solution to the simultaneous inequalities (1) and (2).

We can treat three or more simultaneous inequalities analogously.

The set of all points which satisfy several inequalities is called the **region represented by the simultaneous inequalities.**

Problem 1 Sketch the regions represented by the following sets of simultaneous inequalities:

(1) $\begin{cases} y < 2x + 1 \\ 3y > x - 2 \\ y < -x + 2 \end{cases}$ (2) $\begin{cases} x^2 + y^2 < 4 \\ x + y > 0 \\ x - y < 0 \end{cases}$

Problem 2 What simultaneous inequalities define the first quadrant? What simultaneous inequalities define the second, third, and fourth quadrants?

Demonstration

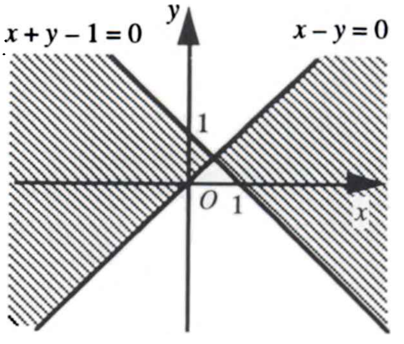
Sketch the region represented by the inequality $(x-y)(x+y-1)>0$.

[Solution]

The given inequality is equivalent to the following pairs of inequalities:

$$\begin{cases} x-y>0 \\ x+y-1>0 \end{cases} \quad \text{or} \quad \begin{cases} x-y<0 \\ x+y-1<0 \end{cases}.$$

Therefore, the region is the union of the regions represented by these two simultaneous inequalities. The figure at the right illustrates this region. The border is not included.



Problem 3

Sketch the region represented by the following inequalities:

(1) $(x-1)(x+2y)>0$

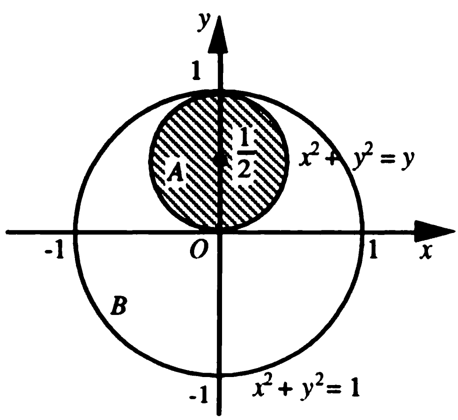
(2) $(x+y+1)(x-2y+4)\le 0$

2 Application of Regions Represented by Inequalities

Regions and Necessary Conditions and Sufficient Conditions

Demonstration 1 Is $x^2 + y^2 \leq y$ a necessary condition or a sufficient condition for $x^2 + y^2 \leq 1$? Use the region represented by each inequality to decide.

[Solution] Let A be the region represented by $x^2 + y^2 \leq y$, and let B be the region represented by $x^2 + y^2 \leq 1$. A is the set of points on the circumference and inside a circle with a radius of $\frac{1}{2}$ and center at $(0, \frac{1}{2})$. B is the set of points on the circumference and inside a circle with a radius of 1 and center at the origin. It is clear from the figure that $A \subseteq B$. The paired values x, y that satisfy $x^2 + y^2 \leq y$ always satisfy $x^2 + y^2 \leq 1$ as well. So the following implication holds:



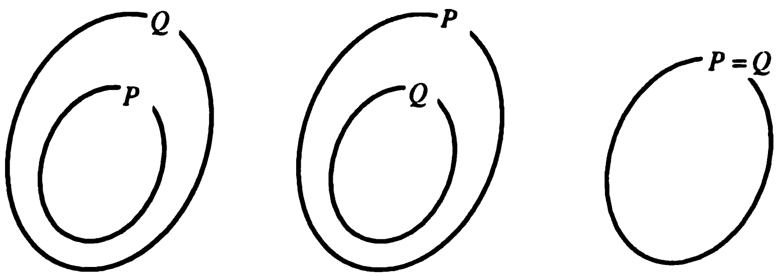
$$x^2 + y^2 \leq y \Rightarrow x^2 + y^2 \leq 1.$$

However, $B \subseteq A$ is not true, so the converse does not hold. Therefore, $x^2 + y^2 \leq y$ is a sufficient condition, but not a necessary condition, for $x^2 + y^2 \leq 1$.

In general, suppose we are given conditions p, q that apply to the points in a plane, and we take P as the set of points that satisfy condition p and Q as the set of points that satisfy condition q. Then the fact that p is a sufficient condition for q while p is a necessary condition for q can be expressed by the following notation using sets P and Q:

> (1) p is a sufficient condition for q: $P \subseteq Q$.
>
> (2) p is a necessary condition for q: $Q \subseteq P$.
>
> Therefore,
>
> (3) p is a necessary and sufficient condition for q:
>
> $P = Q$.

(1) (2) (3)



Problem 1 Is p a necessary or a sufficient condition for q in (1) and (2)?

(1) p: $x \leq 1, y \leq 1$ $\quad$ q: $x^2 + y^2 \leq 1$

(2) p: $x^2 + 2x + y^2 < 0$ $\quad$ q: $(x + 2)^2 + 2(x + 2) + y^2 > 0$

Regions and Minimum and Maximum Values

Demonstration 2 Sketch region D, represented by the following simultaneous inequalities:

$$x \leq 2,\ \ y \leq 2,\ \ x + y \geq 2.$$

Then find the maximum and minimum values of

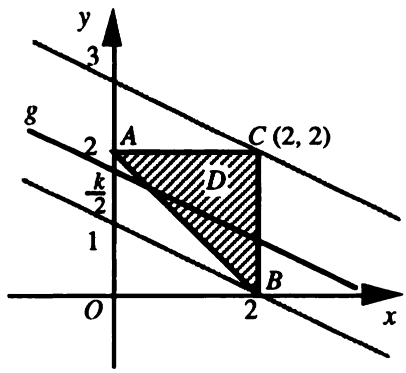
$$z = x + 2y$$

as the point (x, y) moves in this region.

[Solution] Region D is the perimeter and interior of a triangle whose vertices are

$A(0, 2)$, $B(2, 0)$, $C(2, 2)$

as illustrated to the right.



Let's consider a straight line $x + 2y = k$, where k is a constant, and designate this line as g.

If g has a common point with D, then z has the value k at this point. Therefore, we can find the range of the values of z by simply finding the range of the value of the constant k when line g and region D have common points.

Line g has a slope of $-\frac{1}{2}$ and a y-intercept of $\frac{k}{2}$, and g moves parallel to its original position as k varies. As you can see from the figure above, when g passes through point C its y-intercept is 3, and when g passes through point B its y-intercept is 1. Therefore, the condition for g and D to have a common point is:

$$1 \leq \frac{k}{2} \leq 3.$$

Thus, $$2 \leq k \leq 6.$$

Hence, the maximum value of z is 6 and the minimum value is 2.

Problem 2 Find the maximum and minimum values of $2x + y$ as the point (x, y) moves in the region represented by the following inequalities:

$$x + 1 \geq 0,\ \ y + 1 \geq 0,\ \ x - 2y + 2 \geq 0,\ \ x + y \leq 2.$$

Problem 3 Find the maximum and minimum values of $y - x$ if x and y satisfy the inequality $x^2 + y^2 \leq 4$.

Exercises

1. Sketch the region represented by the following inequalities:

   (1) $x + 2y - 4 > 0$

   (2) $1 < x^2 + y^2 < 4$

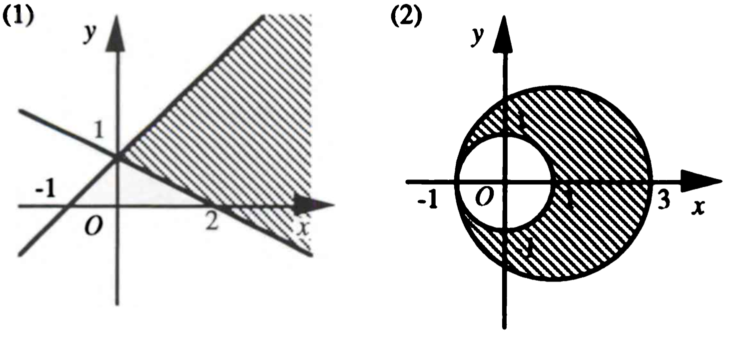
   (3) $x^2 + y^2 \leq 2x + 3$

   (4) $x^2 + y^2 < 9, \quad x + y > 2$

   (5) $0 < x < 1, \quad 0 < y < 1, \quad x^2 + y^2 > 1$

   (6) $(x + y)(x - y) < 0$

2. Find the inequalities which represent the shaded areas in the figures below. Assume that the borders are not included.



3. Find the maximum and minimum values of $x - y$ as the point (x, y) moves in the region represented by the simultaneous inequalities $y \leq 3x$, $2y \geq x$, $x + 3y - 5 \leq 0$.

4. Find the maximum and minimum values of $2x + y$ if x and y satisfy $x^2 + y^2 \leq 4$.

Chapter Exercises

A

1. If we let M and N be the midpoints of diagonals AC and BD of quadrilateral $ABCD$, then the following equality holds:

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2.$$

Prove this.

2. Find the equation of the perpendicular bisector of line segment AB for $A(2, 3)$ and $B(4, 5)$.

3. The equations of lines l_1 and l_2 are $3x + 4y - 18 = 0$ and $x - 2y + 4 = 0$, respectively. Find the value of the constant k if the line $y = kx$ passes through the intersection of l_1 and l_2.

4. Find the value of the constant a if the three points $(a, a + 4)$, $(-2, 6)$, and $(7, 5)$ all lie on the same line.

5. Find the area of the triangle bounded by the three lines $x + y - 7 = 0$, $3x - 2y - 1 = 0$, and $x - 4y + 3 = 0$.

6. Find the radius and the coordinates of the center of a circle through the three points $(8, 4)$, $(3, -1)$, and $(6, 8)$.

7. Given two points $A(4, -2)$ and $B(2, 5)$. Let point P move along the circumference of the circle $x^2 + y^2 = 9$. Find the locus of the centroid G of ΔABP.

8. Find the equation of a line through the point $(1, 5)$ and tangent to the circle $x^2 + y^2 = 1$.

9. Represent the interior of a triangle with vertices $(2, 0)$, $(-3, 0)$, and $(0, 4)$ by means of inequalities.

10. The point (x, y) moves in the region represented by $x \geq 0$, $y \geq 0$, $x + 2y \leq 5$, and $2x + y \leq 4$. Find the maximum value of $4x + 3y$ and the value of x and y for which $4x + 3y$ has a maximum value.

B

1. Find the value of the constant a if the two lines $ax + 4y = 1$ and $x + (a - 3)y = 2$ are parallel.

2. Find the coordinates of a point symmetric to the point $A(3, 2)$ with respect to the line $y = 3x - 2$.

3. Find the range of values of the constant a if the line $ax + y + 1 = 0$ and the circle $x^2 + y^2 + 4x + 3 = 0$ have common points. Find the value of a if they are tangent.

4. Given two fixed points $A(2, 0)$ and $B(0, 1)$. Find the locus of point P satisfying

$$AP : PB = 2 : 1.$$

5. Let Q and R be the points at which two lines are tangent to the circle $x^2 + y^2 = r^2$ if both lines pass through the point $P(a, b)$ outside the circle. Demonstrate that the equation of the line QR is $ax + by = r^2$.

6. Find the slope of a straight line through the point (2, 0), if the length of a chord cut from this line by the circle $x^2 + y^2 = 1$ is 1.

7. Find the coordinates of the points at which the two circles $x^2 + y^2 - 4y - 4 = 0$ and $x^2 + y^2 - 2x = 0$ intersect.

8. Determine the range of the value of r such that the two circles $(x - 4)^2 + (y - 3)^2 = 4$ and $x^2 + y^2 = r^2$ have no common points.

9. Sketch the region represented by the following simultaneous inequalities:

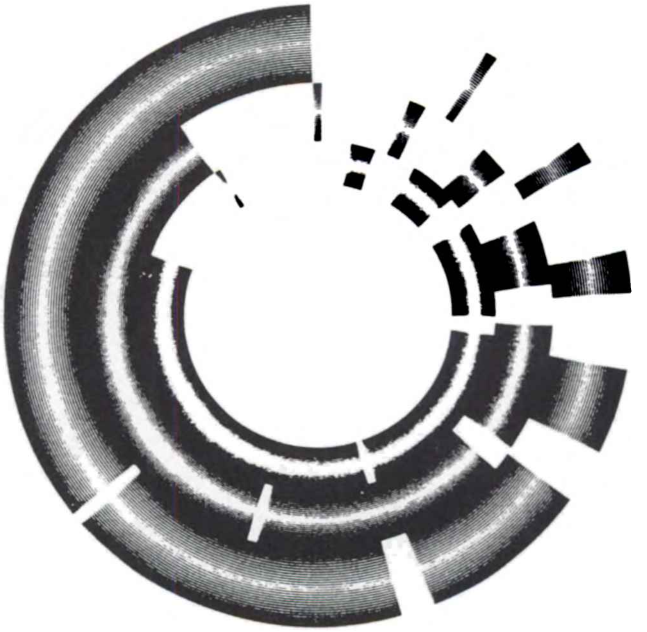
$$\begin{cases} x^2 + y^2 + 2x - 2y < 0 \\ (x + 3y - 2)(x - 2y + 3) > 0 \end{cases}.$$

10. Are conditions (1) and (2) below necessary or sufficient conditions for $x^2 + y^2 \le 2$?

(1) $|x| \le 1$ and $|y| \le 1$ (2) $|x| \le 1$ or $|y| \le 1$

CHAPTER 4

FUNCTIONS



SECTION 1. QUADRATIC FUNCTIONS
SECTION 2. SIMPLE FRACTIONAL FUNCTIONS AND IRRATIONAL FUNCTIONS

Mathematics adopted "change and motion" as an object of study back in the time of the Renaissance, when the watchword was the liberation of humanity. It was then that the concept of the function was born. It represented a great leap forward from Greek mathematics, with the geometry of Euclid (330? – 275? B.C.) at its pinnacle, where the ideal was to find beauty in the harmony of stillness and stability.

The concept of the function was a cornerstone for the invention of differential and integral calculus by such men of genius as Isaac Newton (1642 – 1727). The concept then stimulated the development of the exact sciences centered around physics.

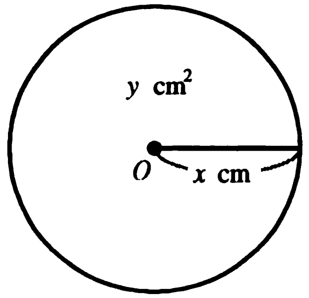
Today, the concept of the function has been extended into the more abstract concept of mapping, imbued with the spirit of geometry, and it performs the leading role as the basis for modern mathematics.

1 QUADRATIC FUNCTIONS

2 Functions and Graphs

Example 1 If we take y cm^2 as the area of a circle of radius x cm, then x and y stand in the relation

$$y = \pi x^2. \qquad (1)$$



If we substitute a positive value for x at (1), then we obtain a single value for y, depending on the substitution. For example,

for $x = 2, y = \pi \times 2^2 = 4\pi$;

for $x = 5, y = \pi \times 5^2 = 25\pi$.

Suppose there are two variables x and y. If a single value of y is determined by each value of x, y is called a **function** of x. And x is referred to as the **independent variable**, and y as the **dependent variable**.

Generally, the fact that y is a function of x is expressed by the symbols $y = f(x)$, $y = g(x)$ (or whatever letter is convenient). For example, if we express the function in Example 1 as $y = f(x)$, then

$$f(x) = \pi x^2.$$

The value of y corresponding to $x = a$ is called the value of the function $y = f(x)$ for $x = a$, and is designated by $f(a)$.

For example, if $y = f(x)$ represents the function in Example 1, then $f(2) = 4\pi$ and $f(5) = 25\pi$.

Problem 1 Find the following values for the function $f(x) = x^2 - 2x + 3$:

(1) $f(1)$ (2) $f(0)$ (3) $f(-3)$ (4) $f(a)$

The set of all the values of the independent variable x for a function $y = f(x)$ is called the **domain** of the function.

For example, for the function $y = \pi x^2$ in Example 1, the variable x represents the radius of any circle, and so the domain of this function is the set of all positive numbers. We can indicate this fact explicitly by writing

$$y = \pi x^2 \quad (x > 0).$$

However, in general, when $f(x)$ is given in terms of x and there is no special stipulation, the domain of the function $y = f(x)$ is usually taken as the set of all values of x such that $f(x)$ is defined.

Example 2 The domain of the functions

$$y = -2x + 3, \quad y = x^2 + 1$$

is the set of all real numbers in both cases. The domain of the function

$$y = \frac{1}{x - 1}$$

is $\{x \mid x \neq 1\}$.

For the function $y = f(x)$, when x takes on all the values in the domain, the set of all the values of $f(x)$ is called the **range** of the function.

Example 3 The range of the function $y = x^2$ is $\{y \mid y \geq 0\}$.

The range of the function $y = x^2$ $(x \geq 2)$ is $\{y \mid y \geq 4\}$.

Problem 2 Give the range of the following functions:

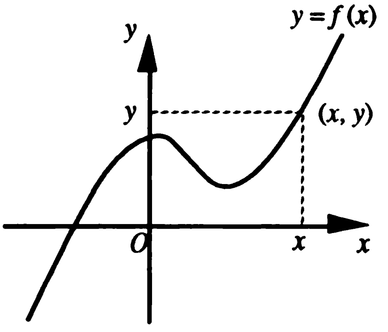
(1) $y = -2x + 3$ (2) $y = x^2 - 1$ (3) $y = (x - 1)^2 + 2$

Problem 3 What is the range of the function $y = \pi x^2$ $(x > 0)$ in Example 1?

Problem 4 Give the range of the following functions:

(1) $y = -2x + 3$ $(x < 0)$ (2) $y = x^2 - 1$ $(x \geq 1)$

If we find the value of the function for each x in the domain of the function $y = f(x)$ and plot the points with coordinates (x, y) in the coordinate plane, the set of points will make up a figure on the plane. This figure is called the **graph** of the function $y = f(x)$.

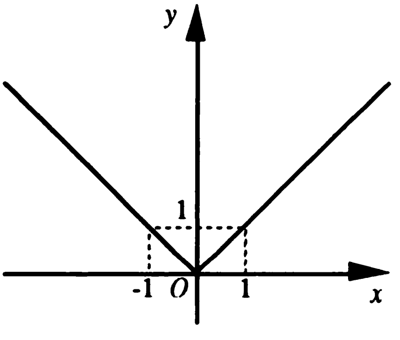


In other words, the graph of the function $y = f(x)$ represents the following set of points in the coordinate plane:

$\{(x, y) \mid y = f(x), x \in \text{domain}\}$.

Topic for Enrichment: The Function $y = |x|$

graph of $y = |x|$



For the function $y = |x|$:

for $x \geq 0$, $y = x$,

for $x < 0$, $y = -x$.

Therefore, the graph of the function is the figure to the right.

The domain of this function is the set of all real numbers, and the range is $\{y \mid y \geq 0\}$.

2 Quadratic Functions and Their Graphs

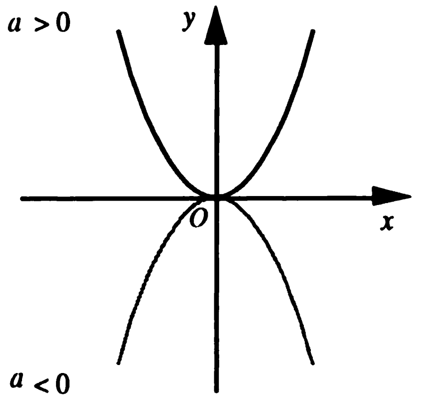
Let's consider the general case of a quadratic function

$$y = ax^2 + bx + c$$

by reviewing the properties of the quadratic function $y = ax^2$ which you learned in junior high school.

The Graph of $y = ax^2$

This graph passes through the origin and is symmetric with respect to the y-axis. Therefore, we can learn about the entire graph by examining only the portion for $x > 0$.



For $a > 0$, if the positive value of x increases, then the values of y increase as well. If the value of x is infinitely great, y also has a positive, infinitely great value.

For $a < 0$, if the positive value of x increases, then the value of y decreases. If the value of x is infinitely great, then y has a negative infinite value.

The graph in the figure above is a curve called a **parabola**. The y-axis is referred to as the **axis** of the parabola $y = ax^2$, and the origin is called its **vertex**. Moreover, this parabola bears one more label:

for $a > 0$, the parabola is **convex down**;

for $a < 0$, the parabola is **convex up**.

Problem 1 Graph the function $y = ax^2$ for the following value of the constant a:

$$1, \quad 2, \quad 3, \quad \frac{1}{2}, \quad -\frac{1}{2}, \quad -1, \quad -2$$

The Graph of $y = a(x - p)^2 + q$

Let's consider how to translate the quadratic function

$$y = ax^2. \tag{1}$$

As you have already learned on page 159, if we translate the figure represented by the equation

$$f(x, y) = 0$$

p units along the x-axis and q units along the y-axis, we obtain a new figure given by the equation

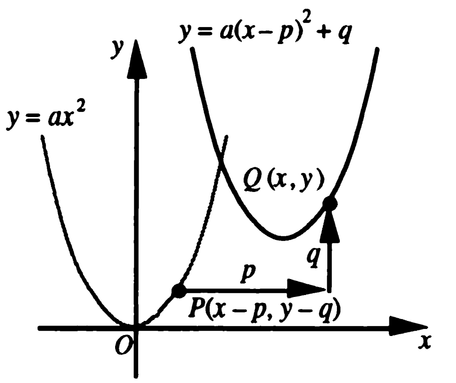
$$f(x-p, y-q) = 0.$$

Therefore, the equation of the figure created by translating the graph of (1) p units along the x-axis and q units along the y-axis is

$$y-q = a(x-p)^2.$$

In other words, that figure is the graph of the function

$$y = a(x-p)^2 + q.$$

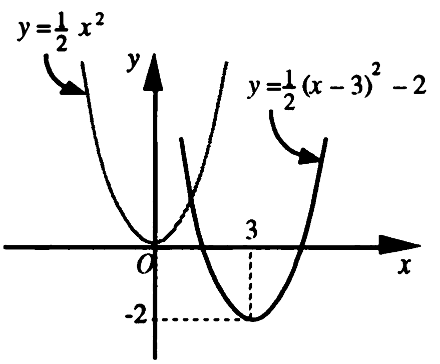


The Graph of $y = a(x-p)^2 + q$

The graph of the quadratic function $y = a(x-p)^2 + q$ is the parabola created by translating the graph of $y = ax^2$ by p units along the x-axis and q units along the y-axis.

The coordinates of the vertex are (p, q), and the equation of the axis is $x = p$.

Example 1 Since the graph of $y = \frac{1}{2}(x-3)^2 - 2$ is the parabola created by translating the graph of $y = \frac{1}{2}x^2$ by 3 units along the x axis and -2 units along the y axis, it takes the form shown in the figure.



The coordinates of the vertex are (3, -2), and the equation of the axis is $x = 3$.

Problem 2 Graph the following quadratic functions:

(1) $y = \frac{1}{2}x^2 - 3$ (2) $y = -3(x+1)^2$

(3) $y = \frac{1}{3}(x+2)^2 - 1$ (4) $y = -\frac{2}{3}(x-1)^2 + 4$

The Graph of $y = ax^2 + bx + c$

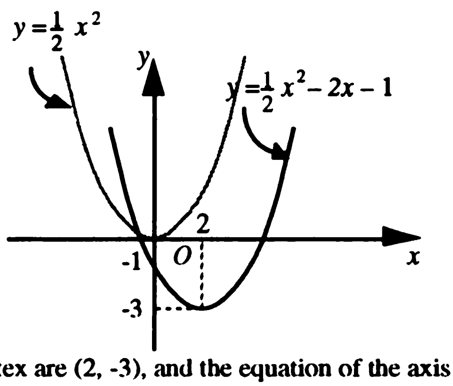
We can graph the quadratic function $y = ax^2 + bx + c$ by transforming it into the form $y = a(x-p)^2 + q$.

Demonstration 1 Graph the function $y = \frac{1}{2}x^2 - 2x - 1$.

[Solution]

$$\begin{aligned} y &= \frac{1}{2}x^2 - 2x - 1 \\ &= \frac{1}{2}(x^2 - 4x) - 1 \\ &= \frac{1}{2}(x-2)^2 - 3 \end{aligned}$$

Therefore, the graph takes the form shown in the figure.



The coordinates of the vertex are (2, -3), and the equation of the axis is $x = 2$.

Problem 3 Graph the following quadratic functions by transforming them into the form $y = a(x-p)^2 + q$. Find the coordinates of the vertex and the equation of the axis.

(1) $y = x^2 + 2x + 3$ (2) $y = -2x^2 + 8x + 1$

(3) $y = 2x^2 + 3x$

In general, a quadratic function $y = ax^2 + bx + c$ can be transformed into

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

Problem 4 Check the above transformation.

We can now formulate the following generalization.

The Graph of $y = ax^2 + bx + c$

The graph of a quadratic function $y = ax^2 + bx + c$ is the parabola created by translating the graph of $y = ax^2$ by $-\frac{b}{2a}$ units along the x-axis and $-\frac{b^2 - 4ac}{4a}$ units along the y-axis.

The coordinates of the vertex are $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$.

The equation of the axis is $x = -\frac{b}{2a}$.

Problem 5 Find the coordinates of the vertex and the equation of the axis of the following functions by using the above formula. Then graph them.

(1) $y = 5x^2 + 2x - 3$ (2) $y = 2x - \frac{1}{2}x^2$

The Maximum and Minimum Values of Quadratic Functions

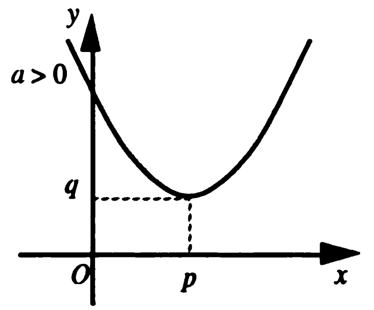
We can now find the maximum and minimum values of a quadratic function based on what we have already learned.

Given the quadratic function

$$y = a(x - p)^2 + q.$$

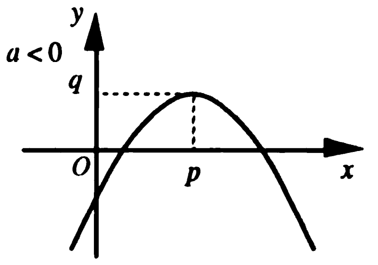
(i) For $a > 0$:

y takes on its minimum value q at $x = p$; there is no maximum value.



(ii) For $a < 0$:

y takes on its maximum value q at $x = p$; there is no minimum value.



We can formulate the following generalization about the general quadratic function $y = ax^2 + bx + c$ by transforming it into

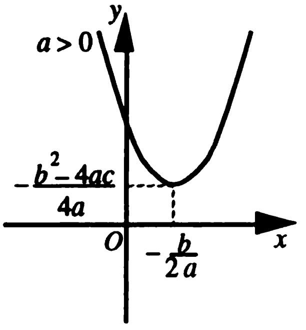
$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

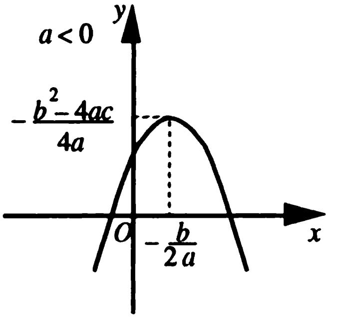
The Maximum and Minimum Values of
$y = ax^2 + bx + c$

Given a quadratic function $y = ax^2 + bx + c$.

(i) For $a > 0$, y takes on its minimum value $-\frac{b^2 - 4ac}{4a}$ at $x = -\frac{b}{2a}$, and it has no maximum value.

(ii) For $a < 0$, y takes on its maximum value $-\frac{b^2 - 4ac}{4a}$ at $x = -\frac{b}{2a}$, and it has no minimum value.





Example 2 (1) Since the quadratic function $y = x^2 - 2x + 3$ can be transformed into

$$y = (x - 1)^2 + 2$$

and the coefficient of the quadratic term is positive, it takes on its minimum value 2 for $x = 1$. It has no maximum value.

(2) Since the quadratic function $y = -2x^2 + 3x + 1$ can be transformed into

$$y = -2(x - \frac{3}{4})^2 + \frac{17}{8}$$

and the coefficient of the quadratic term is negative, it takes on its maximum value $\frac{17}{8}$ for $x = \frac{3}{4}$. It has no minimum value.

Problem 6 Find the maximum and minimum values of the following quadratic functions. Then give the value of x for which they take on those values.

(1) $y = x^2 - 6x + 2$ (2) $y = \frac{1}{2}x^2 + x$

(3) $y = -2x^2 + 3x$ (4) $y = 4 - 2x - \frac{1}{2}x^2$

Demonstration 2 Find the maximum and minimum values of the quadratic function

$$y = x^2 - 2x - 2$$

if its domain is $\{x \mid -1 \leq x \leq 4\}$.

[Solution] If we examine this function for $-1 \leq x \leq 4$, we find that the graph of the function

$$y = x^2 - 2x - 2 = (x - 1)^2 - 3$$

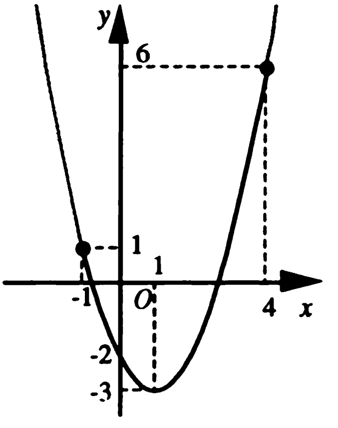
is the solid portion of the curve in the figure. Therefore,

for $x = 1$

y takes on the minimum value -3;

for $x = 4$

y takes on the maximum value 6.



Problem 7 Find the maximum and minimum values of the following functions for the domains given in parentheses. Then find the values of x for which the functions take on those values.

(1) $y = x^2 - 4x \quad (0 \leq x \leq 3)$

(2) $y = x^2 + x - 2 \quad (-2 \leq x \leq 1)$

(3) $y = 5 - x^2 \quad (-1 \leq x \leq 2)$

(4) $y = -x^2 + 3x \quad (-2 \leq x \leq 1)$

Problem 8 We want to make two squares by cutting 60 cm of wire into two pieces and bending them, and we want the sum of the areas of these squares to be a minimum. How must we cut the wire?

3 Equations and Inequalities and the Graphs of Functions

The Graphs of Quadratic Functions and Quadratic Equations

Let's consider how we can solve the quadratic equation

$$ax^2 + bx + c = 0 \qquad (1)$$

using the graph of the quadratic function

$$y = ax^2 + bx + c. \qquad (2)$$

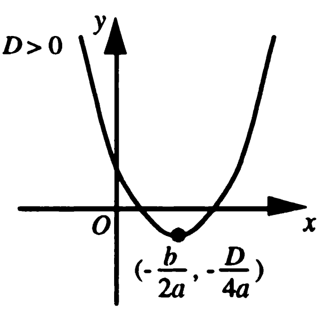
If we let D be the discriminant $b^2 - 4ac$ of (1), then the graph of (2) is a parabola with the vertex $(-\frac{b}{2a}, -\frac{D}{4a})$.

If we assume $a > 0$, then the parabola is convex down and the y-coordinate of the vertex is negative for $D > 0$, equal to 0 for $D = 0$, and positive for $D < 0$.

These facts enable us to draw the following conclusions.

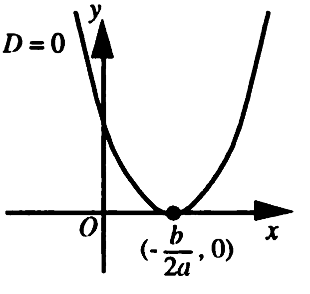
(i) For $D > 0$:

The graph of (2) shows two points shared with the x-axis. The x-coordinates of these points are the two real solutions to quadratic equation (1). In this case, the graph is said to have two intersections with the x-axis.



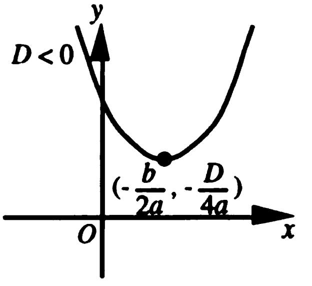
(ii) For $D = 0$:

The graph of (2) shows only one point shared with the x-axis. The x-coordinate of this point is the multiple solution to quadratic equation (1). In this case the graph is said to be **tangent** to the x-axis, and the common point is called the **tangent point**.



(iii) For $D < 0$:

The graph of (2) has no common points with the x-axis. Therefore, (1) has no real solutions.



Problem 1 How will the statement above have to be modified for $a < 0$?

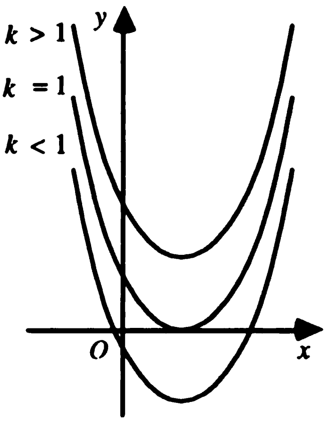
Demonstration 1 How many common points does the parabola $y = x^2 - 2x + k$ have with the x-axis?

[Solution] If we let D be the discriminant of the quadratic equation $x^2 - 2x + k = 0$, then $\frac{D}{4} = 1 - k$.

(i) For $D > 0$, or $k < 1$, the graph shares two points with the x-axis.

(ii) For $D = 0$, or $k = 1$, the graph shares only one point with the x-axis.

(iii) For $D < 0$, or $k > 1$, the graph has no points in common with the x-axis.



Answer: For $k < 1$, 2 points; for $k = 1$, 1 point; and for $k > 1$, no points.

Problem 2 How does the number of common points of the parabola $y = x^2 - kx + 4$ and the x-axis vary depending on the value of k?

We can think analogously about the points shared by the parabola $y = ax^2 + bx + c$ and the line $y = mx + n$.

Since the coordinates of the common points are the real solutions of the simultaneous equations

$$\begin{cases} y = ax^2 + bx + c \\ y = mx + n \end{cases}$$

the real solutions of the quadratic equation

$$ax^2 + (b - m)x + (c - n) = 0$$

–derived by eliminating y from the above simultaneous equations–are the x-coordinates of the common points.

Therefore, the number of common points is identical to the number of real solutions to the quadratic equation.

Problem 3 Determine the number of points shared by the parabola $y = x^2 + 3$ and the following lines:

(1) $y = -2x + n$ (2) $y = mx - 2$

The Graphs of Quadratic Functions and Quadratic Inequalities

Next, let us consider how we can solve quadratic inequalities using the graph of the quadratic function

$$y = ax^2 + bx + c. \tag{1}$$

Here we assume $a > 0$, as before.

If the discriminant D of the quadratic equation

$$ax^2 + bx + c = 0 \tag{2}$$

is positive, let α and β be the non-equal real solutions of (2), and let $\alpha < \beta$. Then the graph of the quadratic function (1) is given in Figures I and II.

Since the solution of the quadratic inequality

$$ax^2 + bx + c > 0$$

is the set of values of x such that $y > 0$ on the graph of (1), the solutions are

$$x < \alpha, \ \beta < x.$$

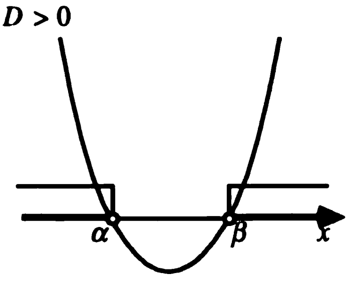


Figure I

Analogously, the solution of the quadratic inequality

$$ax^2 + bx + c < 0$$

is

$$\alpha < x < \beta.$$

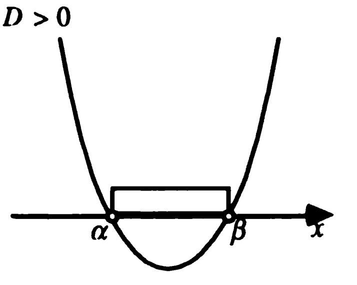


Figure II

Next, for $D = 0$, if we take α as a multiple solution of (2), then the graph of (1) is given in Figure III.

Therefore, the solutions of

$$ax^2 + bx + c > 0$$

are all the real numbers except α.

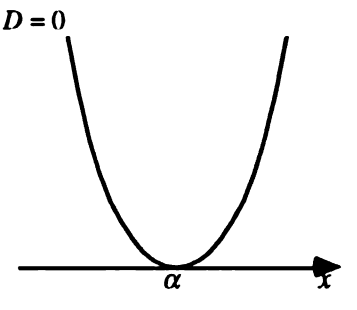


Figure III

Finally, for $D < 0$, Figure IV informs us that the solutions of

$$ax^2 + bx + c > 0$$

are all the real numbers, and

$$ax^2 + bx + c < 0$$

has no solutions.

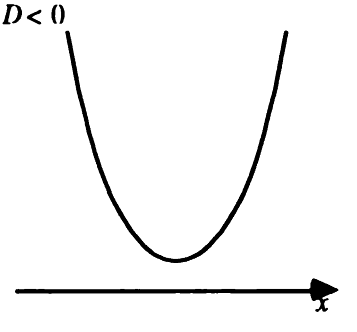


Figure IV

Problem 4 Solve the following quadratic inequalities by considering their graphs:

(1) $x^2 + 4x + 1 < 0$ (2) $3x^2 + 7x + 3 \geq 0$

(3) $9x^2 - 30x + 25 > 0$ (4) $2x^2 - 3x + 4 < 0$

(5) $-x^2 + 5x - 7 < 0$ (6) $x^2 - 4x + 4 \leq 0$

Problem 5 Prove the following statement by means of the graph of the inequality.

For $a > 0$, the necessary and sufficient condition such that for any real number x, the inequality

$$ax^2 + bx + c > 0$$

holds true, is $D = b^2 - 4ac < 0$.

Application of Graphs

Demonstration 2 Find the range of values for the constant k such that the quadratic equation $x^2 + 2(2k + 1)x + (k + 1) = 0$ has two non-equal positive solutions.

[Solution] The condition for this equation to have two non-equal positive solutions is that

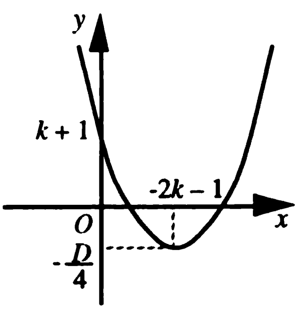
the parabola $y = x^2 + 2(2k + 1)\,x + (k + 1)$

intersects the positive half of the x-axis at two points.

This parabola is convex down, and the coordinates of the vertex are

$$(-2k - 1, -\frac{D}{4}),$$

and it intersects the y-axis at $(0, k + 1)$.



Therefore, the above condition is equivalent to the following three simultaneous conditions.

$$-\frac{D}{4} = -\{(2k+1)^2 - (k+1)\} < 0 \qquad (1)$$

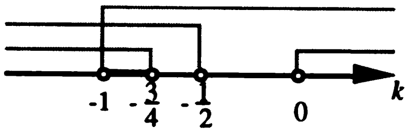
$$-2k - 1 > 0 \qquad (2)$$

$$k + 1 > 0 \qquad (3)$$

From (1), $k < -\frac{3}{4}$, $0 < k$.

From (2), $k < -\frac{1}{2}$.

From (3), $k > -1$.



Thus, the range of the value of k is $-1 < k < -\frac{3}{4}$.

Problem 6 Find the range of values of the constant k such that the quadratic equation $x^2 - kx + 4 = 0$ has two negative solutions.

Topic for Enrichment: Graphs and Inequalities

Let's use graphs to solve the inequality

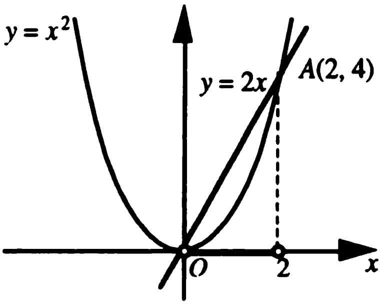
$$x^2 < 2x. \qquad (1)$$

The graphs of

$$y = x^2 \qquad (2)$$

and

$$y = 2x \qquad (3)$$

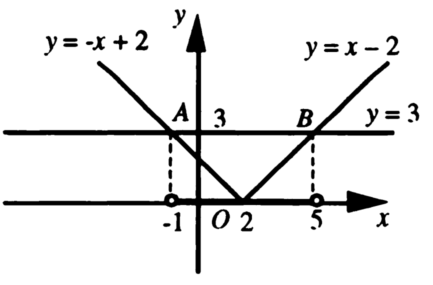


are the parabola and the straight line in the figure to the right. They intersect at the origin O and the point $A(2, 4)$. Therefore, the values of x that satisfy (1), that is, where the graph of (3) lies in the upper half of the graph of (2), is

$$0 < x < 2.$$

Analogously, to solve the inequality

$$|x - 2| < 3$$



using graphs, let's graph $y = |x - 2|$ and $y = 3$. We obtain the graphs in the figure to the right. They intersect at two points A and B. Since point A is where the two lines $y = -x + 2$ and $y = 3$ intersect, its x-coordinate is -1. Since point B is where the two lines $y = x - 2$ and $y = 3$ intersect, its x-coordinate is 5. Therefore, the solution of the inequality is

$$-1 < x < 5.$$

Exercises

1. Find the the quadratic function whose graph is created by translating the quadratic function $y = -3x^2$ such that the vertex lies on the following points:

   (1) (-2, 0) (2) (0, -3)

   (3) (5, 7) (4) (-1, 4)

2. Graph each parabola by finding its vertex and the common points with the coordinate axes.

   (1) $y = 3x^2 - 2x$ (2) $y = (x + 1)(x - 3)$

   (3) $y = -2x^2 - 4x + 6$ (3) $y = \frac{1}{2}x^2 + 2x - 6$

3. Find the quadratic function whose graph is a parabola that satisfies the following conditions:

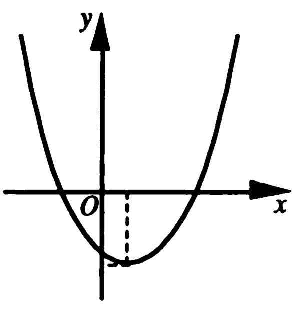
   (1) The coordinates of the vertex are (3, -1), and it passes through the point (6, 0).

   (2) It passes through the three points (-1, -3), (0, 1), and (1, 3).

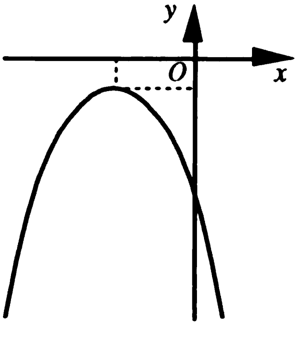
   (3) It passes through the two points (1, 1) and (4, 4), and the vertex lies on the x-axis.

4. The parabolas below are the graphs of the quadratic function $y = ax^2 + bx + c$. Determine the signs of a, b, c, and $D = b^2 - 4ac$ for both figures.

   (1)



   (2)



5. Find the maximum and minimum values of the following quadratic functions for the domain in parentheses.

(1) $y = 3x - x^2$ $(-1 \leq x \leq 2)$ (2) $y = x^2 + 5x + 4$ $(-3 \leq x \leq 0)$

6. Determine how many points are shared by the parabola $y = x^2 + 2kx + 1$ and the line $y = 2x - 3$.

2 SIMPLE FRACTIONAL FUNCTIONS AND IRRATIONAL FUNCTIONS

1 Fractional Functions and Their Graphs

When y, a function of x, is a fractional expression in x, such as

$$y = \frac{4}{x}, \quad y = -\frac{1}{2x+3}, \quad y = \frac{3x+4}{x-2}$$

y is said to be a **fractional function** of x.

The domain of a fractional function excludes the value of the variable for which the denominator would be equal to 0.

For example, the domains of the three functions above are

$$\{x \mid x \neq 0\}, \quad \{x \mid x \neq -\frac{3}{2}\}, \quad \{x \mid x \neq 2\}.$$

The Graph of $y = \frac{a}{x}$

You have already learned the function $y = \frac{a}{x}$, expressing an inverse proportion, in junior high school.

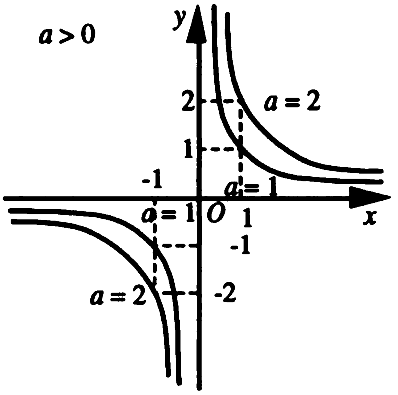
The figures on the next page give the graphs for various values of a.

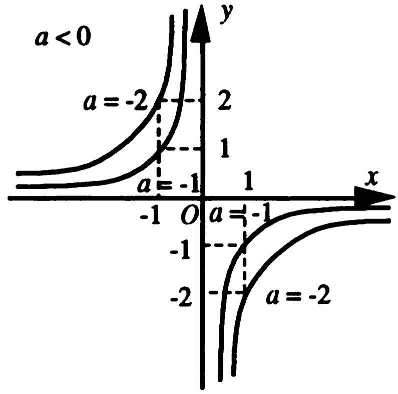
For $a > 0$:

Each of the two curves consists of one part in the first quadrant and one part in the third quadrant.

For $a < 0$:

Each of the two curves consists of one part in the second quadrant and one part in the fourth quadrant.





As these curves move away from the origin, they approach closer and closer to the x- and y-axes. A straight line which a curve approaches infinitely closely is called the **asymptote** of the curve. Two perpendicular lines are the asymptotes for the curve $y = \frac{a}{x}$. Such a curve is called a **rectangular hyperbola**.

Problem 1 Graph the function $y = \frac{4}{x}$. Find the coordinates of the points at which this curve intersects the line $y = x$.

Problem 2 Graph the function $y = -\frac{3}{x}$. Find the coordinates of the points at which this curve intersects the line $y = -x$.

The Graph of $y = \frac{a}{x - p} + q$

If we translate the graph of the function $y = \frac{a}{x}$ by p units along the x-axis and q units along the y-axis, then the equation of the new figure is

$$y - q = \frac{a}{x - p}.$$

Therefore,

$$y = \frac{a}{x - p} + q.$$

Moreover, this same translation also moves the x- and y-axes to

$$x = p, \quad y = q.$$

Hence we can formulate the following general statement:

The Graph of $y = \dfrac{a}{x - p} + q$

The graph of $y = \dfrac{a}{x - p} + q$ is a rectangular hyperbola created by translating the graph of $y = \dfrac{a}{x}$ by p units along the x-axis and q units along the y-axis.

The asymptotes are the two lines $x = p$ and $y = q$.

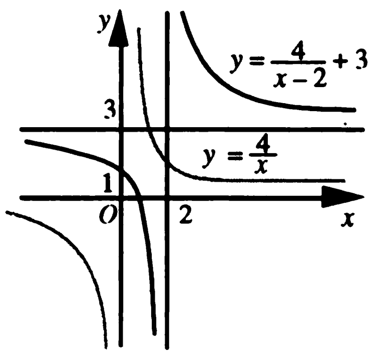
Example The graph of $y = \dfrac{4}{x - 2} + 3$ is created by translating the graph of $y = \dfrac{4}{x}$

by 2 units along the x-axis;

by 3 units along the y-axis.

The equations of the asymptotes are

$x = 2$ and $y = 3$.



Problem 3 Graph the following functions. Find the equations of their asymptotes.

(1) $y = \dfrac{2}{x - 3}$ (2) $y = \dfrac{3}{x + 2} - 4$

(3) $y = -\dfrac{4}{x - 2} + 5$

Problem 4 Graph y, a function of x, determined by the following equations:

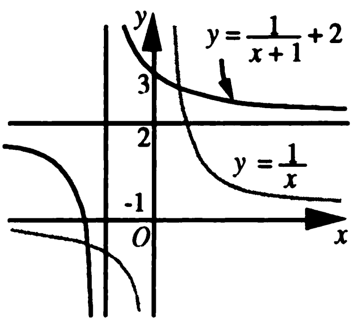
(1) $(x + 1)\,y = -1$ (2) $(x - 1)(y + 2) = 4$

The Graph of $y = \dfrac{ax + b}{cx + d}$

Demonstration Graph the function $y = \dfrac{2x + 3}{x + 1}$.

[Solution] If the numerator of the right side is divided by the denominator, the quotient is 2 with a remainder of 1. Therefore, we can carry out the following transformation:

$$y = \frac{1}{x + 1} + 2.$$



Thus, the graph of this function is created by translating the graph of $y = \dfrac{1}{x}$ by -1 units along the x-axis and 2 units along the y-axis.

We get the graph in the figure above. The asymptotes are the two lines

$$x = -1, \qquad y = 2.$$

Problem 5 Graph the following functions. Then find the equations of their asymptotes.

(1) $y = \dfrac{3x}{x - 1}$ (2) $y = \dfrac{2x - 7}{x - 2}$

(3) $y = \dfrac{1 - 2x}{x + 2}$ (4) $y = \dfrac{x - 1}{3 - x}$

2 Irrational Functions and Their Graphs

If y, a function of x, is expressed in a form with x under a radical sign, as in

$$y = \sqrt{x}, \quad y = 3\sqrt{2-x}, \quad y = \sqrt{x^2+1} + 5$$

y is called an **irrational function** of x.

The domain of an irrational function is the set of values for the variable which do not make the expression under the radical sign negative.

For example, the domains of the three functions above are

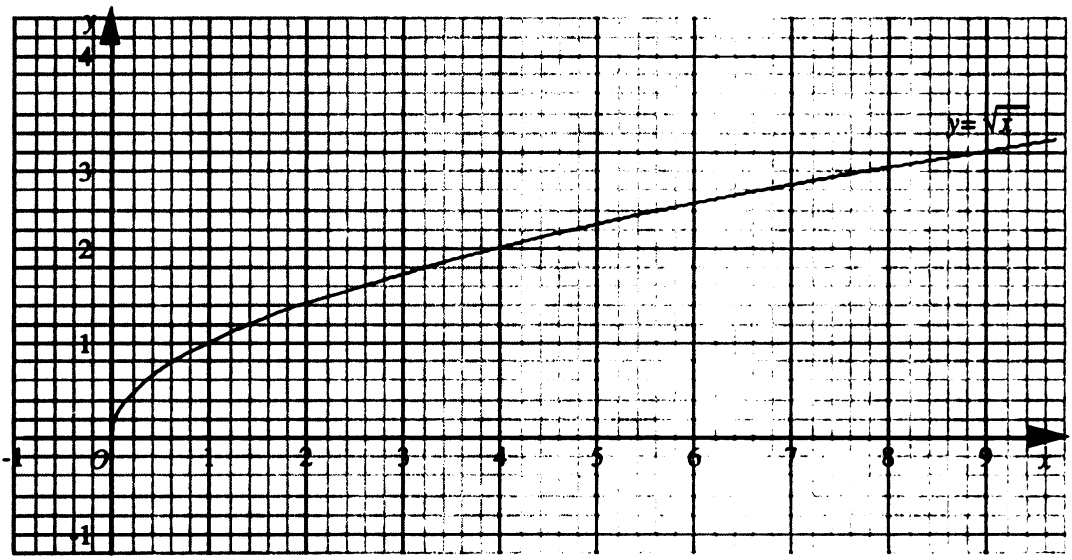
$\{x \mid x \geq 0\}$, $\{x \mid x \leq 2\}$, and the set of all real numbers.

The Graph of $y = \sqrt{ax}$

Let's consider the graph of

$$y = \sqrt{x} \quad (x \geq 0).$$

First we find the value of y corresponding to various values of x from the square root table, and plot the points with the coordinates (x, y) on the coordinate plane, and then we obtain the curve in the figure below. This is a graph of $y = \sqrt{x}$.



Problem 1 Check that the point $(x, \sqrt{x})$ lies on the curve in the figure above for $x = 0, 1, 4,$ and 9.

Let's consider the graph of $y = \sqrt{x}$ in more detail.

The condition which specifies that the point (x, y) lies on the graph of $y = \sqrt{x}$ is

$$y^2 = x, \quad y \geq 0. \tag{1}$$

(1) means that the point (x, y) lies on the curve represented by the equation $y^2 = x$, and that it is located on or above the x-axis.

First, let's consider the curve represented by the equation

$$y^2 = x. \tag{2}$$

Interchanging x and y in (2), we obtain the equation

$$y = x^2. \tag{3}$$

The curve represented by (3) is a parabola convex down with the origin as its vertex, as you have already learned.

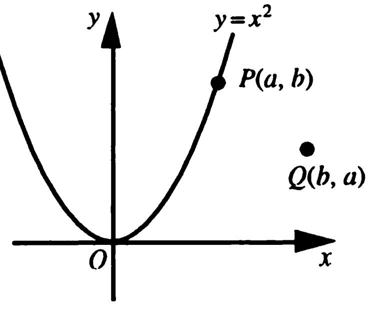
If we let the point $P(a, b)$ be any point on the parabola represented by equation (3), then we have

$$b = a^2.$$

Rewriting this equality as

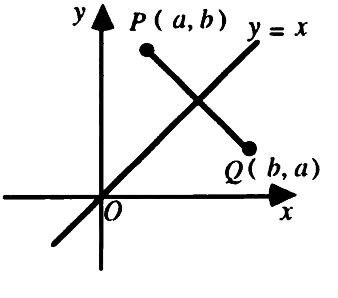
$$a^2 = b$$

we find that the point $Q(b, a)$ lies on the curve represented by equation (2).



Conversely, if the point $Q(b, a)$ lies on the curve represented by (2), then the point $P(a, b)$ lies on the parabola represented by (3).

Points $P(a, b)$ and $Q(b, a)$ are symmetric with respect to the line $y = x$, as you learned on page 139.



Therefore, the curves represented by the equation

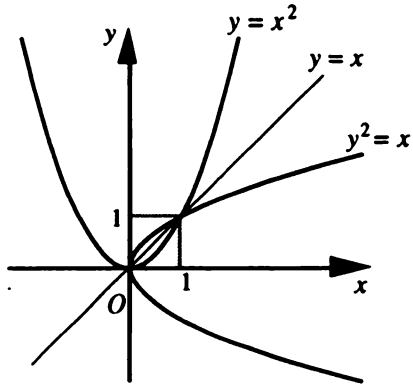
$$y^2 = x$$

and the parabola

$$y = x^2$$

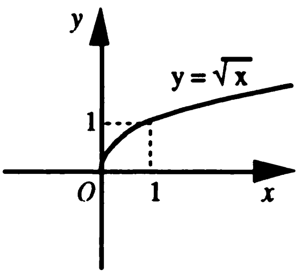
are symmetric with respect to the line $y = x$.

These facts enable us to formulate the following general statement:



The Graph of $y = \sqrt{x}$

The graph of $y = \sqrt{x}$ is the portion of the parabola created by reflecting the parabola $y = x^2$ with respect to the line $y = x$ corresponding to $y \geq 0$.



Demonstration 1 Graph the irrational function $y = \sqrt{-2x}$.

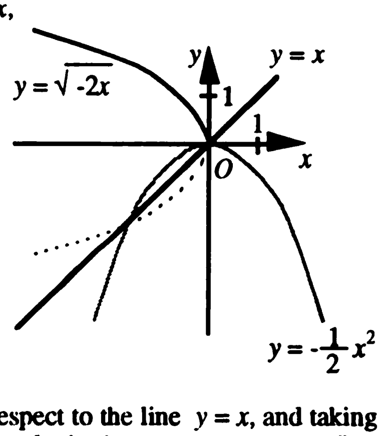
[Solution] Since $y = \sqrt{-2x} \Leftrightarrow y^2 = -2x$, we see that $y \geq 0$.
Interchanging x and y in

$$y^2 = -2x,$$

we obtain

$$x^2 = -2y.$$

Thus, $y = -\frac{1}{2}x^2$.

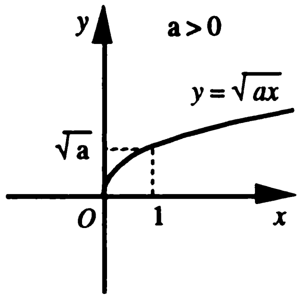


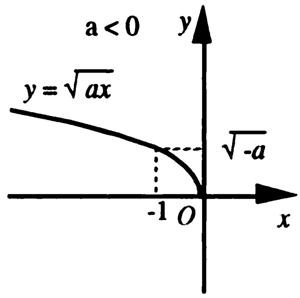
By reflecting this graph with respect to the line $y = x$, and taking the part corresponding to $y \geq 0$, we obtain the graph we want to find.

Problem 2 Graph the following irrational functions as in Demonstration 1.

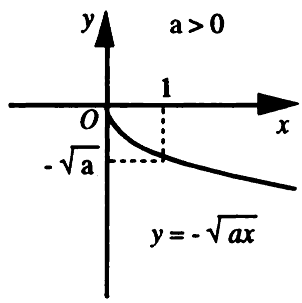
(1) $y = \sqrt{2x}$ (2) $y = \sqrt{-\frac{1}{2}x}$

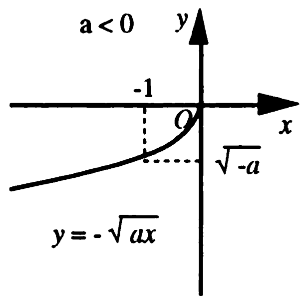
In general, the graph of $y = \sqrt{ax}$ takes the form illustrated by the following figures, depending on the sign of a. Both figures are the portions of a parabola corresponding to $y \geq 0$, where the x-axis is the axis of the parabola.





The graph of $y = -\sqrt{ax}$ is created by reflecting the graph of $y = \sqrt{ax}$ with respect to the x-axis.





Problem 3 Graph the following functions:

(1) $y = -2\sqrt{x}$ (2) $y = -\sqrt{-x}$

The Graph of $y = \sqrt{ax + b}$

The equation of the curve created by translating the graph of

$$y = \sqrt{ax}$$

by p units along the x-axis and q units along the y-axis is

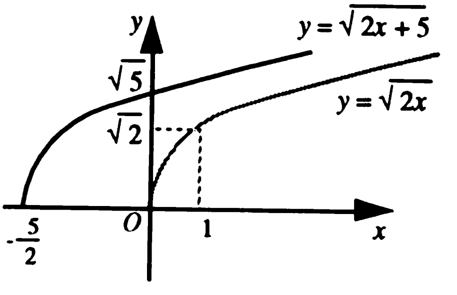
$$y - q = \sqrt{a(x - p)}\ .$$

Demonstration 2 Graph $y = \sqrt{2x + 5}$.

[Solution] Since $y = \sqrt{2x+5}$ can be transformed into

$$y = \sqrt{2(x + \frac{5}{2})}\ ,$$

the graph is created by translating the graph of $y = \sqrt{2x}$ by $-\frac{5}{2}$ units along the x-axis.



Problem 4 Graph the following functions:

(1) $y = \sqrt{x - 3}$ (2) $y = \sqrt{2 - x}$

(3) $y = \sqrt{2x - 3}$ (4) $y = \sqrt{-5 - 2x}$

You can translate the graph of $y = -\sqrt{ax}$ analogously.

Problem 5 Graph the following functions:

(1) $y = -\sqrt{x - 1}$ (2) $y = -\sqrt{2x + 1}$

Demonstration 3 Find the coordinates of the points at which the graph of $y = \sqrt{2x+5}$ intersects the line $y = \frac{1}{2}x$.

[Solution] The x-coordinates of the intersections of the two graphs are the values of x that satisfy

$$\sqrt{2x+5} = \frac{1}{2}x. \qquad (1)$$

Squaring both sides of (1), we get

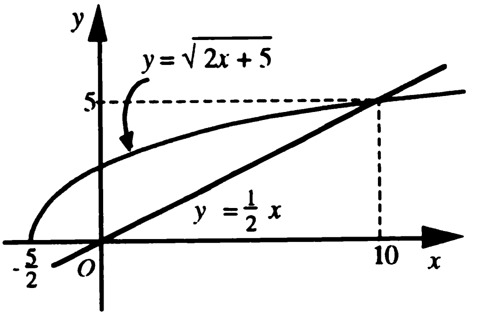
$$2x + 5 = \frac{1}{4}x^2.$$

That is, $x^2 - 8x - 20 = 0$.

Solving this equation, we obtain

$$x = -2,\ 10.$$

From the graph to the right, we can see that $x = 10$ is the correct value, since we know that $x > 0$.



Thus, $y = 5$.

Therefore, the coordinates of the intersection are $(10, 5)$.

Problem 6 Find the coordinates of the points at which the following pairs of functions intersect:

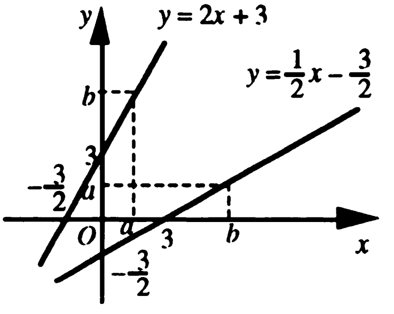
(1) $\begin{cases} y = \sqrt{x-2} \\ y = \frac{1}{3}x \end{cases}$ (2) $\begin{cases} y = -\sqrt{x} \\ y = x - 1 \end{cases}$

1 Inverse Functions

Example 1 Both the domain and range of the function $f(x) = 2x + 3$ are the set of all real numbers. If we take b as any real number, then a real number a which satisfies

$$b = f(a), \qquad \text{that is,} \qquad b = 2a + 3$$

has the single value $a = \frac{1}{2}b - \frac{3}{2}$.



Therefore, if we let

$$g(x) = \frac{1}{2}x - \frac{3}{2},$$

then we have

$$b = f(a) \iff a = g(b).$$

In other words, for the function $f(x) = 2x + 3$, the relation between the value of the variable x and the corresponding value of the function is the inverse of the analogous relation for the function $g(x) = \frac{1}{2}x - \frac{3}{2}$.

Example 2 The range of $f(x) = x^2 + 1$, with the set $A = \{x \mid x \geq 0\}$ as its domain, is the set $B = \{y \mid y \geq 1\}$. If we take b as any element of B, then a, an element of A that satisfies

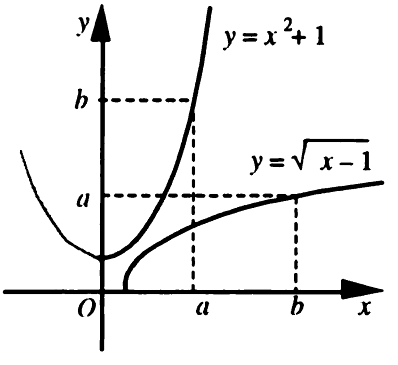
$$b = f(a),$$

that is,

$$b = a^2 + 1,$$

has the single value

$$a = \sqrt{b-1}.$$



Therefore, if we let

$$g(x) = \sqrt{x-1} \quad (x \geq 1),$$

then for a, an element of A, and b, an element of B, we get

$$b = f(a) \iff a = g(b).$$

In general, if the domain of a function $y = f(x)$ is A and the range is B, and a, an element of A that satisfies $b = f(a)$, has a single value corresponding to each element b, an element of B, then there exists a function $y = g(x)$ which satisfies properties (1) and (2) below.

(1) The domain of $y = g(x)$ is B and the range is A.

(2) For a, an element of A, and b, an element of B, the following implication holds:

$$b = f(a) \iff a = g(b).$$

This function $y = g(x)$ is called the **inverse function** of $y = f(x)$.

For example, for Example 1 and Example 2 above:

the inverse function for $y = 2x + 3$ is $y = \frac{1}{2}x - \frac{3}{2}$;

the inverse function for $y = x^2 + 1 \quad (x \geq 0)$ is $y = \sqrt{x - 1}$.

Condition for an Inverse Function

The function $y = f(x)$ has an inverse function if a, which satisfies $b = f(a)$, is assigned a single value in the domain corresponding to each b in the range.

The definition of an inverse function gives us the following property.

If a function $y = f(x)$ has an inverse function, the domain and range of $y = f(x)$ are interchanged for its inverse $y = g(x)$.

Problem 1 Which of the following functions has an inverse?

(1) $y = \frac{1}{2}x - 3$ (2) $y = x^2$ (3) $y = x^2 \ (x \leq 0)$

Finding an Inverse Function

We can find the inverse function

$$y = \frac{1}{2}x - \frac{3}{2}$$

for the function

$$y = 2x + 3 \tag{1}$$

by the following method:

(a) Consider function (1) as an equation and solve it for x to obtain

$$x = \frac{1}{2}y - \frac{3}{2}.$$

(b) Interchange x and y in the result yielded by (a).

Demonstration 1 Find the inverse of the function $y = -\frac{1}{3}x + 4$.

[Solution] Solving $y = -\frac{1}{3}x + 4$ for x,

$$x = -3y + 12.$$

Interchanging x and y,

$$y = -3x + 12.$$

This is the inverse function we want to find.

Problem 2 Find the inverse of the function $y = -3x + 12$.

In general, the inverse of an inverse function is the original function.

Demonstration 2 Find the inverse of the quadratic function

$$y = x^2 + 1 \tag{1}$$

with a domain of $\{x \mid x \le 0\}$. Find the domain of the inverse function.

[Solution] Solving $y = x^2 + 1$ $(x \le 0)$ for x gives us

$$x = -\sqrt{y - 1}\ .$$

Therefore, the inverse function for (1) is

$$y = -\sqrt{x - 1}\ . \tag{2}$$

Since the range of function (1) is $\{y \mid y \ge 1\}$, the domain of the inverse function (2) is $\{x \mid x \ge 1\}$.

Problem 3 Find the inverse function for each of the following functions, and give their domains.

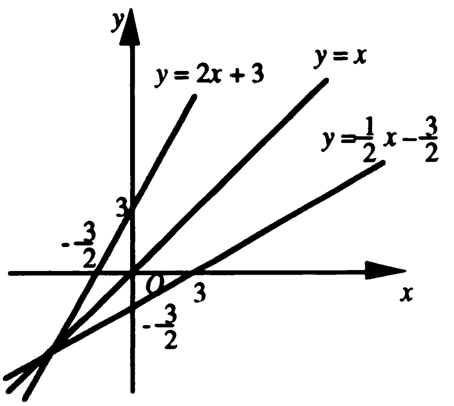
(1) $y = \frac{1}{3}x + 2$ (2) $y = 3x^2 \quad (x \ge 0)$

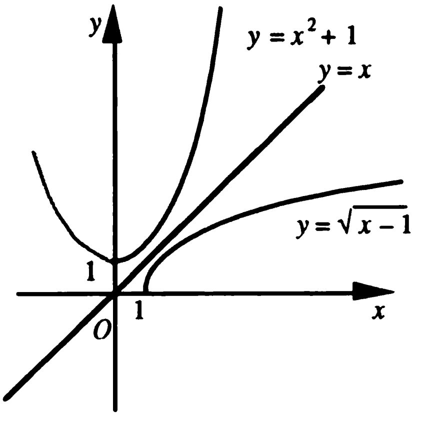
(3) $y = x^2 - 2 \quad (x \le 0)$ (4) $y = -\sqrt{x}$

(5) $y = \frac{1}{2x}$ (6) $y = \frac{1}{x} + 1$

The Graphs of Inverse Functions

The graphs of the function $y = 2x + 3$ and its inverse $y = \frac{1}{2}x - \frac{3}{2}$ or the graphs of the function $y = x^2 + 1$ $(x \geq 0)$ and its inverse $y = \sqrt{x - 1}$ are symmetric with respect to the line $y = x$, as illustrated in the figures below.

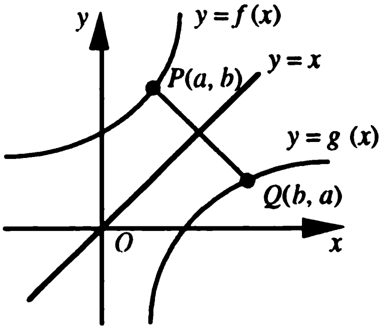




In general, if a function $y = f(x)$ has an inverse function $y = g(x)$, we have the following implication:

$$b = f(a) \Leftrightarrow a = g(b).$$

Therefore, the fact that point $P(a, b)$ lies on the graph of the function $y = f(x)$ is equivalent to the fact that point $Q(b, a)$ lies on the graph of the inverse function $y = g(x)$. And point $P(a, b)$ and point $Q(b, a)$ are symmetric with respect to the line $y = x$.



> The graph of the function $y = f(x)$ and the graph of its inverse $y = g(x)$ are symmetric with respect to the line $y = x$.

Problem 4 Verify the above statement for the graphs of the function $y = x^2 + 1$ $(x \leq 0)$ and its inverse $y = -\sqrt{x - 1}$ in Demonstration 2.

Problem 5 Graph the following functions and their inverses:

(1) $y = x^2 - 1$ $(x \geq 0)$ (2) $y = \frac{2}{x} + 1$ $(x > 0)$

Exercises

1. **Find the functions whose graphs are created by translating a rectangular hyperbola $y = \frac{2}{x}$:**

   (1) -5 units along the x-axis;

   (2) -4 units along the y-axis;

   (3) 3 units along the x-axis and 5 units along the y-axis.

2. **Graph the following functions. Find the equations of their asymptotes.**

   (1) $y = \frac{4}{x + 3}$ (2) $y = 4 - \frac{2}{x + 1}$

   (3) $y = \frac{4}{2x - 3}$ (4) $y = \frac{4x}{2x + 3}$

3. **Find the intersection of the rectangular hyperbola $y = \frac{1}{x}$ and the line $y = \frac{5}{2} - x$.**

4. **Find the functions whose graphs are created by moving the graph of $y = 2\sqrt{x}$ by:**

   (1) reflecting it about the x-axis.

   (2) reflecting it about the y-axis.

   (3) translating it -2 units along the x-axis.

5. **Graph the following functions:**

   (1) $y = \sqrt{3x - 5}$ (2) $y = -\sqrt{4 - x}$

6. **Find the inverse for each of the following functions $f(x)$. Find the domains of the inverse functions.**

   (1) $f(x) = \frac{2}{3}x + 5$ (2) $f(x) = \frac{1}{x + 2} + 1$

   (3) $f(x) = \sqrt{1 - x}$ (4) $f(x) = -\sqrt{x + 4}$

Chapter Exercises

A

1. Find the coordinates of the vertices of the following quadratic functions. Find the functions whose graphs are created by translating the graphs of the following functions so that the vertices lie on the point (2, - 4).

   (1) $y = 3x^2 - 6x + 2$ (2) $y = x - \frac{1}{2}x^2$

2. Find the quadratic functions whose graphs are parabolas that satisfy the following conditions:

   (1) Passes through the two points (0, 1) and (3, -2), and the x-coordinate of the vertex is 1.

   (2) Passes through the point (1, -2) and is tangent to the x-axis at the point (-1, 0).

3. Find the value of p and q such that the function $y = x^2 + px + q$ takes on a minimum value for $x = 4$, and the value $y = 5$ for $x = 2$.

4. Find the value of k such that the parabola $y = 2x^2 + kx + 4$ is tangent to the x-axis.

5. Graph the following functions:

   (1) $y = \dfrac{3x - 4}{x}$ (2) $y = \dfrac{4x}{1 - x}$

6. What function, when graphed, gives the curve created by reflecting the function $y = 3x^2 + 1$, with a domain of $\{x \mid x \le 0\}$, about the line $y = x$?

7. Graph the following functions:

   (1) $y = \sqrt{2x + 10}$ (2) $y = - \sqrt{3 - 2x}$

8. Find the values that a may take so that the parabola $y = a(x - 2)^2 - 3$ intersects the negative ray of the x-axis.

B

1. 50 articles are sold at a price of 60 yen apiece. If the price increases by 10 yen, then the number of articles sold decreases by 5. What price will yield the maximum gross sales receipts?

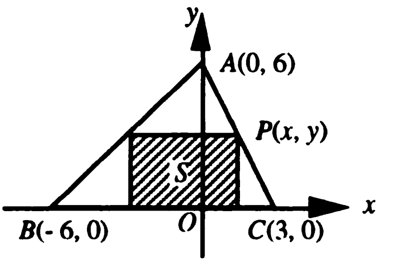
2. Determine the range of values for p such that the quadratic equation $x^2 - 2px + p - 2 = 0$ has one positive solution and one negative solution.

3. The parabola $y = x^2 + 2px + q$ passes through the point (2, 1), and its vertex lies on the x-axis. Find the values of p and q.

4. Point $P(x, y)$ lies on side AC of ΔABC in the figure.

   (1) Express the area S of the shaded rectangle in terms of x.

   (2) Find the maximum value of S and the corresponding coordinates of point P.



5. Determine the range of values for a such that the inequality

$$x^2 + ax + (a + 3) > 0$$

holds true for all real numbers x.

6. If we mix 6% and 8% salt solutions at the ratio of $x : 1$, we obtain a solution of y% salt. Express y as a function of x, and then graph it.

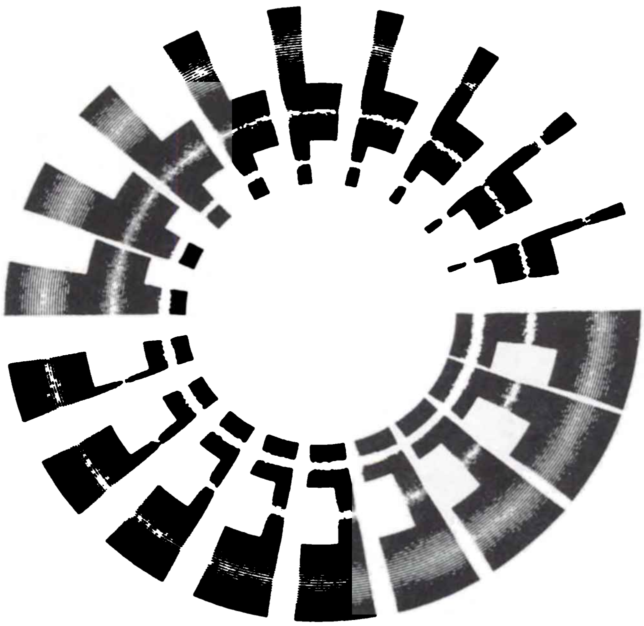
7. Find the inverse of the function $y = \frac{3x - 2}{4 - x}$, with a domain of $\{x \mid x < 4\}$, and then graph it.

8. Determine the number of points at which the graph of the function $y = \sqrt{2x - 1}$ intersects the line $y = x + k$ for the following values of k.

$$-1, \quad -\frac{1}{2}, \quad 0, \quad \frac{1}{2}$$

CHAPTER 5

TRIGONOMETRIC RATIOS



SECTION 1. TRIGONOMETRIC RATIOS
SECTION 2. APPLICATIONS OF TRIGONOMETRIC RATIOS

Ever since the time of the ancient Babylonians and Greeks, triangles have been applied to measurements, astronomical observations, the art of navigation, and so forth.

If one angle in a right triangle other than the right angle is fixed, then the ratios of any two sides of the triangle are determined. This property is the basis for practical applications. The ratio of any two sides of a right triangle forms a trigonometric ratio of that angle.

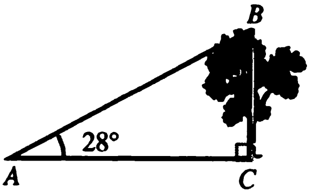
Even today, the concept of a trigonometric ratio is very important for practical applications in measurement, architecture, navigation, and other areas.

1 TRIGONOMETRIC RATIOS

1 Tangent

We can use the following method to find the height of a tree.

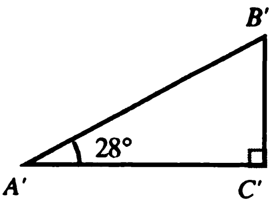
Define point A at some distance from the tree, and measure $\angle A$ and the distance AC. Suppose we have



$$\angle A = 28°,\ AC = 15 \text{ m}.$$

If we draw $\Delta A'B'C'$, a reduced copy of ΔABC, then the two triangles are similar and we know that

$$\frac{BC}{AC} = \frac{B'C'}{A'C'}.$$



Therefore,

$$BC = AC \times \frac{B'C'}{A'C'}.$$

Thus, if we find the value of $\frac{B'C'}{A'C'}$, we can also find the value of BC.

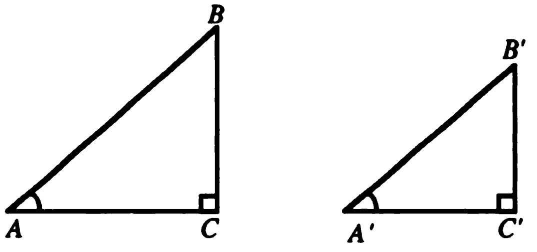
Problem 1 Make a reduced copy $\Delta A'B'C'$ of the above triangle with $A'C' = 5$ cm, and measure $B'C'$. Then find the value of $\frac{B'C'}{A'C'}$. Find the height of the tree BC using this value and $AC = 15$ m.

In general, given two right triangles ABC and $A'B'C'$, where $\angle C$ and $\angle C'$ are right angles, if

$$\angle A = \angle A'$$

then these right triangles are similar.
So we know that

$$\frac{BC}{AC} = \frac{B'C'}{A'C'}.$$



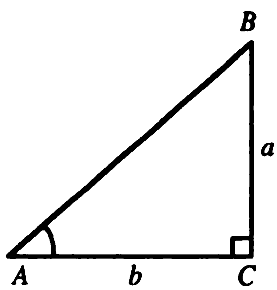
Thus, the value of $\frac{BC}{AC}$ is determined only by A, the measure of $\angle A$.

This ratio $\frac{BC}{AC}$ is referred to as the **tangent** of $\angle A$, and it is designated by **tan** A.

For the right triangle ABC to the right, if we let $BC = a$ and $AC = b$, then

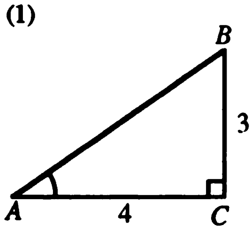
$$\tan A = \frac{a}{b}$$

$$a = b \tan A.$$

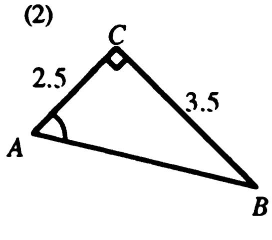


Problem 2 Find the value of the tangent of A in the following figures.

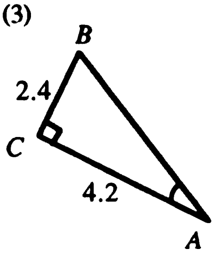
(1)



(2)



(3)



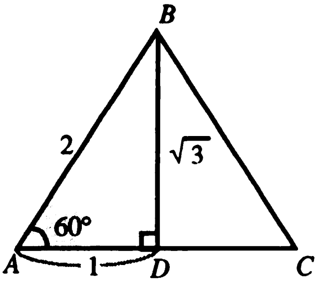
Demonstration Find tan 60° using an equilateral triangle.

[Solution] Draw perpendicular BD from vertex B of equilateral triangle ABC to the opposite side AC.

If we let $AD = 1$, then $AB = 2$. From the Pythagorean theorem we know that

$$BD = \sqrt{2^2 - 1^2} = \sqrt{3}.$$

Therefore, $\tan 60° = \dfrac{BD}{AD} = \sqrt{3}$.



Problem 3 Find tan 30° using the figure in the Demonstration above.

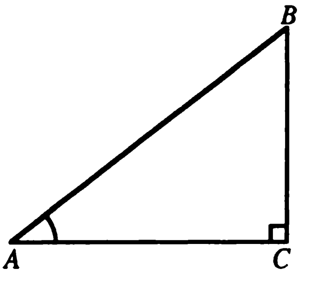
Problem 4 Find tan 45° using an isosceles right triangle.

2 Sine and Cosine

In a right triangle where $\angle C$ is a right angle,

$$\frac{BC}{AB}, \frac{AC}{AB}$$

are determined only by a, the measure of $\angle A$, just as with the tangent.



$\dfrac{BC}{AB}$ is referred to as the **sine**, and is designated as **sin** A.

$\dfrac{AC}{AB}$ is referred to as the **cosine**, and is designated as **cos** A.

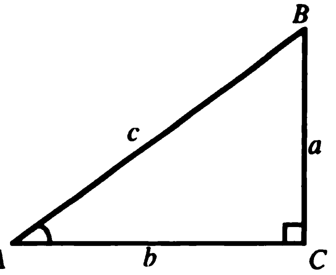
In the right triangle ABC to the right, let us assume that

$$BC = a, AC = b, \text{ and } AB = c.$$

Then

$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}$$

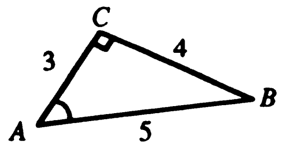
$$a = c \sin A, \quad b = c \cos A.$$



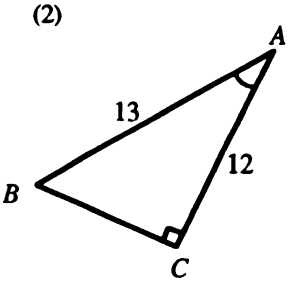
Tangent, sine, and cosine are all called **trigonometric ratios.**

Problem 1 Find the sine and cosine of A in the following figures:

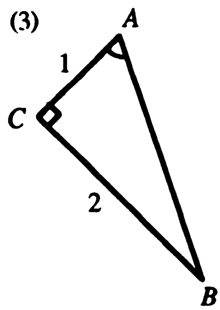
(1)



(2)



(3)



Problem 2 Find the sine and cosine of 30° and 60° using the figure in the Demonstration on the preceding page.

Problem 3 Find the sine and cosine of 45° using an isosceles right triangle.

The values of the trigonometric ratios for angles from 0° to 90° can be found from the table of trigonometric ratios in the Appendix.

Problem 4 Find the following values from the table of trigonometric ratios.

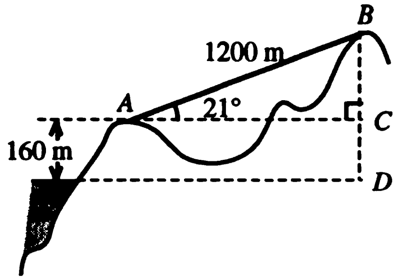
(1) sin 15° (2) cos 53° (3) tan 67°

Problem 5 Find the value of A that satisfies each equation from the table of trigonometric ratios.

(1) $\tan A = 0.1763$ (2) $\sin A = 0.9659$

(3) $\cos A = 0.9659$

Demonstration 1 A ropewalk leads from point A, 160 m above sea level, to mountain top B. It is 1,200 m from A to B along a straight line. AB forms an angle of 21° with the horizontal plane. Find BD, the elevation above sea level of mountain top B.



[Solution] Let us define point C as in the figure. Since

$$BC = AB \sin A$$

$$BC = 1200 \sin 21°$$

$$= 1200 \times 0.3584 \approx 430.$$

Therefore, the elevation above sea level of B is

$$BD = BC + CD$$

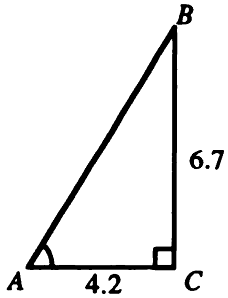
$$\approx 430 + 160 = 590.$$

Answer: approximately 590 m

Note: The symbol $\approx$ means "approximately equal to."

Problem 6 Find AC, the horizontal distance between point A and point B in the above Demonstration.

Demonstration 2 Find A, the measure of $\angle A$, for the right triangle in the figure to the right.



[Solution] $\tan A = \dfrac{BC}{AC} = \dfrac{6.7}{4.2} \approx 1.5952$

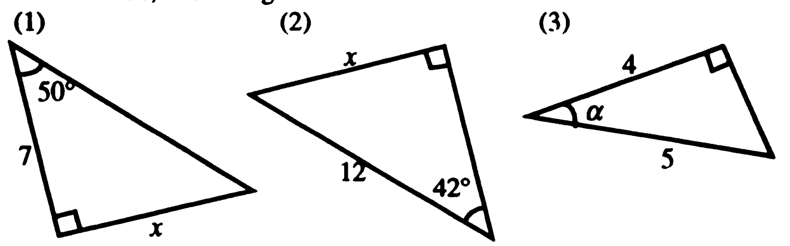
From the table of trigonometric ratios we know that

$$\tan 58° = 1.6003 .$$

Therefore, A is approximately 58°.

Answer: approximately 58°

Problem 7 Find α, the measure of the remaining angle, and x, the length of the indicated side, in each figure:

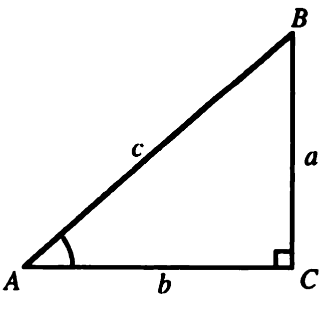


1 Relations among Trigonometric Ratios

Let's examine the relations that hold among the trigonometric ratios.

In the right triangle ABC to the right,

$$a = c \sin A, \quad b = c \cos A.$$



Therefore,

$$\tan A = \frac{a}{b} = \frac{c \sin A}{c \cos A}.$$

Thus, $$\tan A = \frac{\sin A}{\cos A}.$$

Next,

$$(\sin A)^2 + (\cos A)^2 = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2 + b^2}{c^2}.$$

From the Pythagorean theorem we know that $a^2 + b^2 = c^2$, and so

$$(\sin A)^2 + (\cos A)^2 = 1.$$

$(\sin A)^2$ and $(\cos A)^2$ are usually written as $\sin^2 A$ and $\cos^2 A$.
Thus,

$$\mathbf{\sin^2 A + \cos^2 A = 1.}$$

Demonstration Find $\cos A$ and $\tan A$ for right triangle ABC if

$$\sin A = \frac{4}{7}.$$

[Solution] From $\sin^2 A + \cos^2 A = 1$, we know that

$$\cos^2 A = 1 - \sin^2 A.$$

From $\sin A = \frac{4}{7}$,

$$\cos^2 A = 1 - \left(\frac{4}{7}\right)^2 = \frac{33}{49}.$$

Since $\cos A > 0$,

$$\cos A = \sqrt{\frac{33}{49}} = \frac{\sqrt{33}}{7}$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{4}{\sqrt{33}} = \frac{4\sqrt{33}}{33}.$$

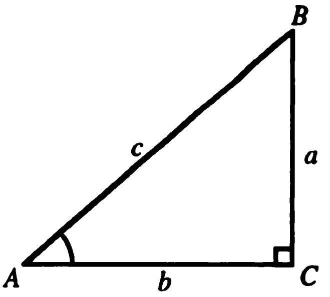
Problem 1 Find $\sin A$ and $\tan A$ for right triangle ABC, if $\cos A = \frac{2}{5}$.

Problem 2 Prove the following equality:

$$1 + \tan^2 A = \frac{1}{\cos^2 A}.$$

In the right triangle to the right,

$$\sin B = \frac{b}{c}, \quad \cos A = \frac{b}{c}.$$



Since $B = 90° - A$,

$$\sin(90° - A) = \cos A. \qquad (1)$$

Analogously,

$$\cos(90° - A) = \sin A. \qquad (2)$$

Problem 3 Check that formula (2) holds.

Problem 4 Prove that the following formula holds:

$$\tan(90° - A) = \frac{1}{\tan A}.$$

Problem 5 Express each trigonometric ratio as a trigonometric ratio of an angle of less that 45°.

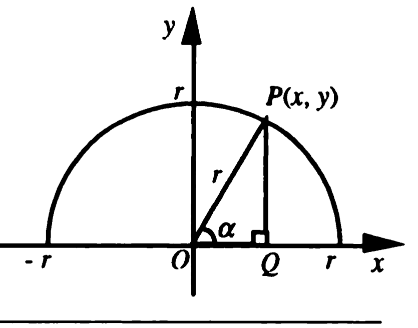
(1) $\sin 56°$ (2) $\cos 82°$ (3) $\tan 65°$

1 Extending Trigonometric Ratios

Until now, we have defined the trigonometric ratios of acute angles using a right triangle. Let's think about trigonometric ratios in terms of coordinates.

Draw a circle with a radius of r and center at the origin.* Then draw radius OP to form an acute angle α with the positive ray of the x-axis. If we draw perpendicular PQ from P to the x-axis, then

$$\sin\alpha = \frac{PQ}{OP}, \quad \cos\alpha = \frac{OQ}{OP}, \quad \tan\alpha = \frac{PQ}{OQ}.$$

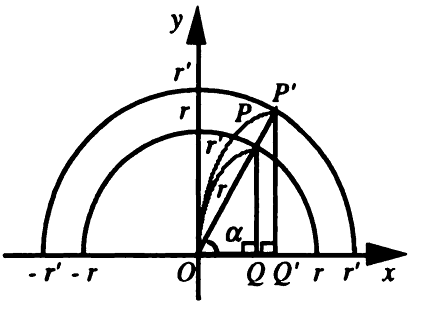


* It is sufficient to draw a semicircle, as in the figure.

Therefore, if we take (x, y) as the coordinates of P, we get

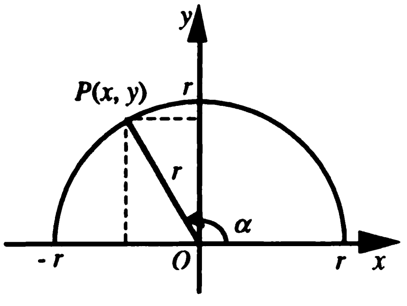
$$\sin\alpha = \frac{y}{r}, \quad \cos\alpha = \frac{x}{r}, \quad \tan\alpha = \frac{y}{x}. \qquad (1)$$

We can see that the value of the ratios in (1) are adequately defined without regard to the radius r of circle O, based on the similarity of ΔOPQ and $\Delta OP'Q'$ in the figure to the right.



Hence, we will define the trigonometric ratios of an angle α for $0° \le \alpha \le 180°$, including an obtuse angle, in the following way.

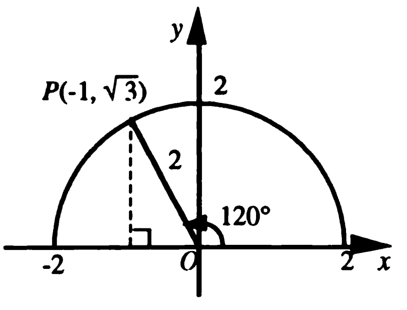
Given a circle with a radius of r and center at the origin, we let OP be the radius that defines an angle α from the positive ray of the x-axis in a counterclockwise direction, and we take (x, y) as the coordinates of point P.



Then

$$\sin\alpha = \frac{y}{r}, \qquad \cos\alpha = \frac{x}{r}, \qquad \tan\alpha = \frac{y}{x}.$$

Example 1 If $\alpha = 120°$, if we draw a figure with a radius of 2, then the coordinates of point P are $(-1, \sqrt{3})$. Therefore, from the definition,



$$\sin 120° = \frac{\sqrt{3}}{2},$$

$$\cos 120° = -\frac{1}{2}, \qquad \tan 120° = -\sqrt{3}.$$

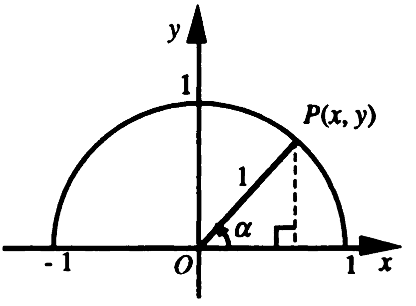
Problem 1 Find the sine, cosine, and tangent of 135° by drawing a circle with a radius of 2, as in Example 1.

Problem 2 What will be the coordinates of point P corresponding to 120° on the circumference, if we draw a circle of radius 1?

The circle with a radius of 1 and center at the origin is called the **unit circle.**

When we work with a unit circle, if we let (x, y) be the coordinates of point P defined by an angle α, then

$$\sin \alpha = y, \qquad \cos \alpha = x.$$



Our new definition gives us the following trigonometric ratios for the special angles $\alpha = 0°, 90°$, and $180°$:

$\sin 0° = 0$	$\sin 90° = 1$	$\sin 180° = 0$
$\cos 0° = 1$	$\cos 90° = 0$	$\cos 180° = -1$
$\tan 0° = 0$	$\tan 90°$ – no value exists	$\tan 180° = 0$

Problem 3 Verify the facts above using the unit circle.

Problem 4 Check that the following formulas also hold under the extended definition.

(1) $\sin^2 \alpha + \cos^2 \alpha = 1$ (2) $\tan \alpha = \dfrac{\sin \alpha}{\cos \alpha}$

(3) $1 + \tan^2 \alpha = \dfrac{1}{\cos^2 \alpha}$

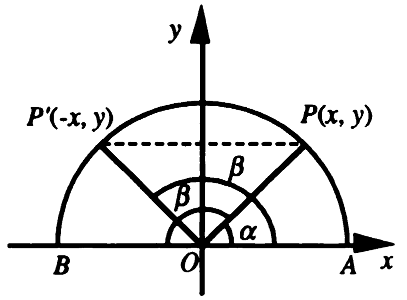
The Trigonometric Ratios of $180° - \alpha$

In the figure to the right, let

$$\angle AOP = \alpha, \quad \angle POB = \beta.$$

Then

$$\beta = 180° - \alpha.$$



If we take point P' such that $\angle AOP' = \beta$, then P and P' are symmetric with respect to the y-axis. Therefore, if we let (x, y) be the coordinates of point P, then the coordinates of point P' are $(-x, y)$. Thus,

$$\sin\beta = \sin\alpha, \quad \cos\beta = -\cos\alpha, \quad \tan\beta = -\tan\alpha.$$

The equality $\beta = 180° - \alpha$ gives us the following formulas:

$$\sin(180° - \alpha) = \sin\alpha$$
$$\cos(180° - \alpha) = -\cos\alpha$$
$$\tan(180° - \alpha) = -\tan\alpha$$

The trigonometric ratios of an obtuse angle can be found by converting them into the trigonometric ratios of an acute angle by means of these formulas.

Example 2 $\sin 120° = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$

Problem 5 Fill in the blanks with the values of the trigonometric ratios.

α	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
$\sin\alpha$									
$\cos\alpha$									
$\tan\alpha$									

Example 3 $\cos 145° = \cos(180° - 35°) = -\cos 35°$

From the table of trigonometric ratios,

$\cos 145° = -\cos 35° = -0.8192$.

Problem 6 Find the sine, cosine, and tangent of the following angles:

(1) $100°$ (2) $160°$

Demonstration Find the angle α that satisfies the following equalities. Here we assume that $0° \le \alpha \le 180°$.

(1) $\sin\alpha = \frac{1}{2}$ (2) $\cos\alpha = -\frac{1}{\sqrt{2}}$

[Solution] (1) On the unit circle, the angle α corresponds to the point with a y-coordinate of $\frac{1}{2}$.

Therefore, the two angles $\angle AOP$ and $\angle AOP'$ in Figure I are the angle α we need to find.

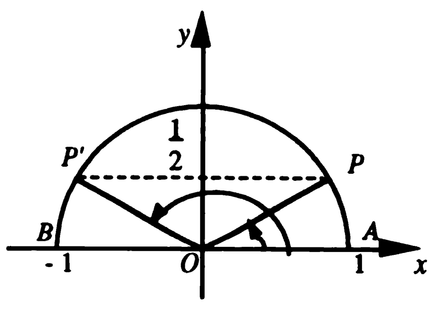


Figure I

Thus, $\alpha = 30°, 150°$.

(2) On the unit circle, the angle α corresponds to the point with an x-coordinate of $-\frac{1}{\sqrt{2}}$.

Therefore, $\angle AOP$ in Figure II is the angle α we want to find.

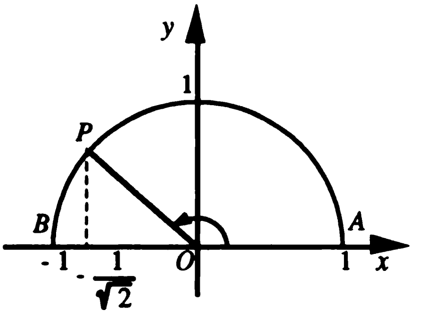


Figure II

Thus, $\alpha = 135°$.

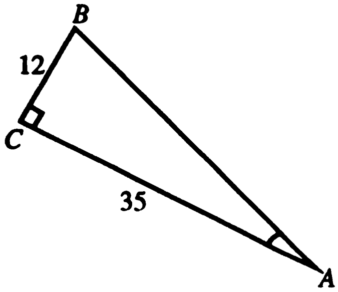
Problem 7 Find angle α that satisfies each equality. Here we assume that $0° \leq \alpha \leq 180°$.

(1) $\sin \alpha = \frac{\sqrt{3}}{2}$ (2) $\tan \alpha = -\sqrt{3}$

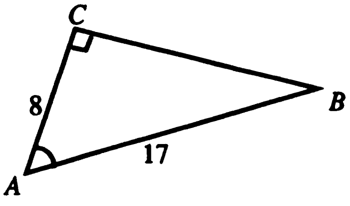
Exercises

1. Find the sine, cosine, and tangent of A for the right triangles below:

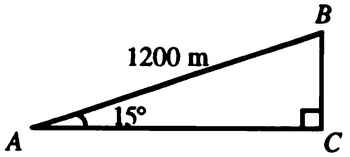
(1)



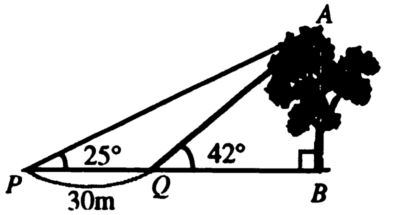
(2)



2. When we move 1,200 m along a slope that forms an angle of 15° with the horizontal plane, what is the distance AC on the horizontal plane? How high is point B above sea level, if point A is 50 m above sea level?



3. When we look up at treetop A from point P, $\angle APB$ is 25°. When we look up at A from point Q, 30 m in front of point P horizontally, $\angle AQB$ is 42°. Find AB, the height of the tree.



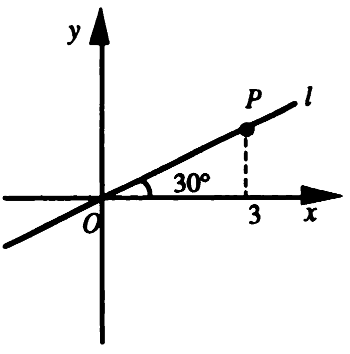
4. Find the following values if α is acute and $\cos \alpha = \frac{5}{13}$.

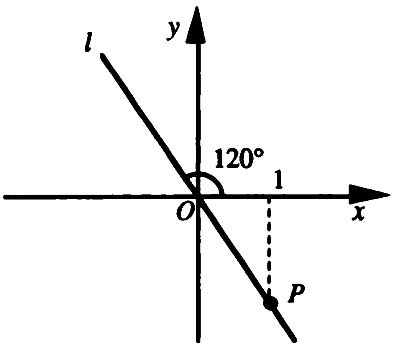
(1) $\sin \alpha$ (2) $\tan \alpha$ (3) $\cos (180° - \alpha)$

5. Find angle α to satisfy the following equalities. Here we assume that $0° \le \alpha \le 180°$.

(1) $\sin \alpha = \frac{1}{\sqrt{2}}$ (2) $\cos \alpha = -\frac{\sqrt{3}}{2}$ (3) $\tan \alpha = -\frac{1}{\sqrt{3}}$

6. Find the y-coordinates of point P on line l in the figures below:

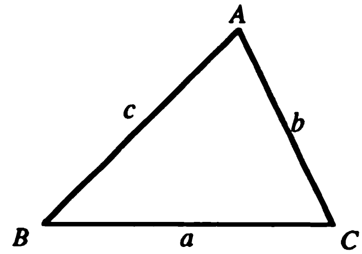




2 APPLICATIONS OF TRIGONOMETRIC RATIOS

Let's consider the relations among the lengths of the sides and the measures of the angles of triangles.

Here we will designate the measures of the three angles in ΔABC as A, B, and C, and we will designate the sides opposite these angles as a, b, and c, respectively.



1 The Sine Theorem

The **Sine Theorem** below relates the sines of angles and the lengths of the opposite sides to the radius of a circumscribed circle.

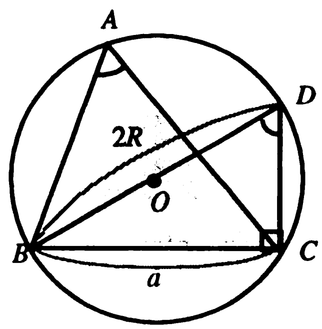
> **Sine Theorem**
>
> $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$
>
> Here, R denotes the radius of the circumscribed circle.

[Proof] First let's prove

$$\frac{a}{\sin A} = 2R. \qquad (1)$$

(i) If A is acute:

Since O, the center of the circle circumscribed around ΔABC, lies inside circle segment BAC, point D, the endpoint of diameter BOD, lies on arc BAC. Therefore,



$$\angle A = \angle BDC.$$

Since

$$\angle BCD = 90°,\ BD = 2R,\ BC = a,$$

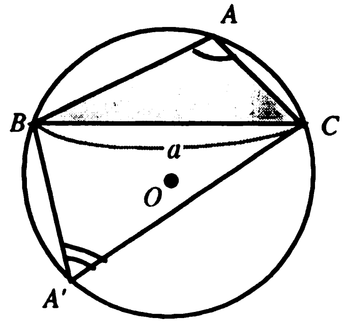
we obtain

$$a = 2R \sin \angle BDC = 2R \sin A.$$

Thus, (1) holds.

(ii) If A is obtuse:

Take point A' on the circumference of the circumcircle outside of chord BAC. Let A' be the measure of $\angle BA'C$. Since



$$A + A' = 180°,$$

A' is an acute angle. Therefore, (i) gives us

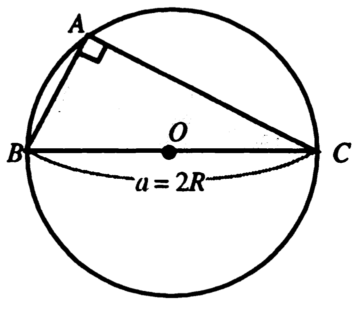
$$\frac{a}{\sin A'} = 2R.$$

Since $\sin A' = \sin (180° - A) = \sin A$,

we can conclude that (1) holds.

(iii) If $A = 90°$:

Since BC, the side opposite to A, is a diameter, we know that



$$a = 2R.$$

Since $\sin A = \sin 90° = 1$,

(1) holds in this case, too.

$\dfrac{b}{\sin B} = 2R,\ \dfrac{c}{\sin C} = 2R$ can be proved analogously.

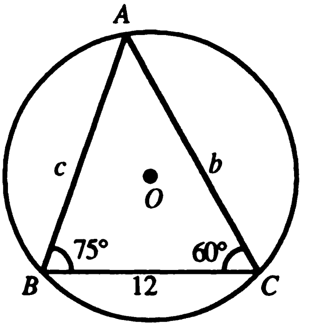
Demonstration Find R, the radius of circumscribed circle of ΔABC, when $a = 12$, $B = 75°$, and $C = 60°$. Find c, too.

[Solution] Since

$$A = 180° - (75° + 60°) = 45°$$

$$a = 12,$$

we obtain



$$2R = \frac{12}{\sin 45°} = \frac{12}{\frac{1}{\sqrt{2}}} = 12\sqrt{2}\,.$$

Therefore, $R = 6\sqrt{2}$.

Next, for $c = 2R \sin C$, $C = 60°$.

$$c = 12\sqrt{2} \sin 60° = 12\sqrt{2} \times \frac{\sqrt{3}}{2} = 6\sqrt{6}$$

Answer: $R = 6\sqrt{2}$, $c = 6\sqrt{6}$

Problem 1 Find the radius of a circle circumscribed around ΔABC if $a = 10$ and $A = 135°$. Find b for $B = 30°$.

Problem 2 What kind of triangle is ΔABC if the following holds?

$$\sin^2 A + \sin^2 B = \sin^2 C$$

The Sine Theorem can be expressed as an extended ratio:

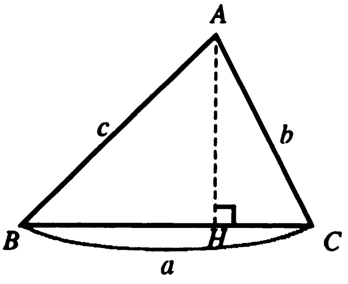
$$a : b : c = \sin A : \sin B : \sin C.$$

Problem 3 Find $a : b : c$ in ΔABC for $A : B : C = 3 : 4 : 5$, using the table of trigonometric ratios.

2 The Cosine Theorem

In ΔABC, where B and C are acute angles, let H be the point at which BC intersects the perpendicular from A to BC.

Since



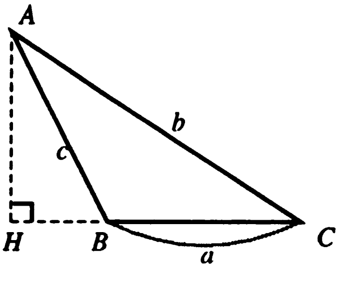
$$BC = BH + HC$$

$$BH = c \cos B$$

$$HC = b \cos C,$$

the following equality holds:

$$a = b \cos C + c \cos B. \qquad (1)$$



Analogously, the following equalities also hold:

$$b = c \cos A + a \cos C$$

$$c = a \cos B + b \cos A.$$

Problem 1 Prove that (1) also holds if B is a right or obtuse angle.

The **Cosine Theorem** below relates the cosines of the angles and the lengths of the sides of a triangle.

Cosine Theorem

$$a^2 = b^2 + c^2 - 2bc \cos A$$

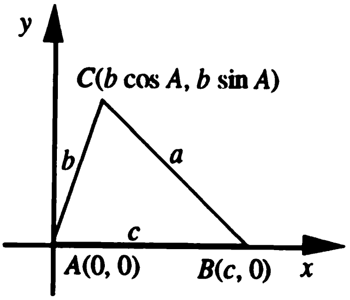
$$b^2 = c^2 + a^2 - 2ca \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

[Proof] If we define coordinate axes for ΔABC as in the figure to the right, the coordinates of the three vertices A, B, and C are

$(0, 0)$, $(c, 0)$,

$(b\cos A, b\sin A)$.



Since a is the distance between the two points B and C,

$$\begin{aligned} a^2 &= (b\cos A - c)^2 + (b\sin A)^2 \\ &= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A \\ &= b^2 + c^2 - 2bc\cos A. \end{aligned}$$

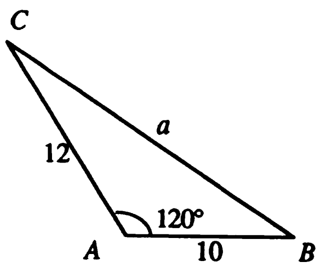
We obtain the other two equations analogously.

Demonstration 1 Find a in ΔABC, for

$A = 120°$,

$b = 12$,

$c = 10$.



[Solution] Since $a^2 = b^2 + c^2 - 2bc\cos A$, we get

$$\begin{aligned} a^2 &= 12^2 + 10^2 - 2 \times 12 \times 10 \times \cos 120° \\ &= 144 + 100 + 120 = 364. \end{aligned}$$

Therefore, $a = 2\sqrt{91}$.

Problem 2 Find a in ΔABC for $A = 60°$, $b = 4$, $c = 6$.

Problem 3 Find A in ΔABC for $a = 7$, $b = 3$, $c = 5$.

Problem 4 Prove the following implications for ΔABC:

(1) If A is acute, then $a^2 < b^2 + c^2$.

(2) If A is obtuse, then $a^2 > b^2 + c^2$.

Demonstration 2 What kind of triangle is ΔABC, if the following equality holds:

$$b \cos A = a \cos B.$$

[Solution] From the Cosine Theorem

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \qquad \cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

Substituting these two expressions in both sides of the equality and rearranging, we obtain

$$b^2 + c^2 - a^2 = c^2 + a^2 - b^2.$$

Therefore, $2(b^2 - a^2) = 0.$

$$2(b - a)(b + a) = 0$$

Since $a + b > 0$,

$$b - a = 0.$$

Thus, $a = b$.

Answer: An isosceles triangle with $a = b$

3 The Area of a Triangle

The area S of ΔABC can be found from the following formula.

Given Two Sides and the Angle between Them

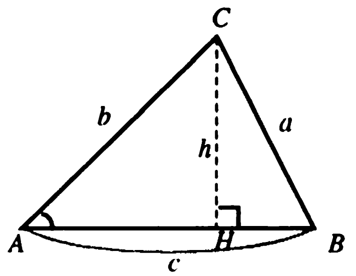
Formula for Two Sides and the Angle between Them

$$S = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C$$

[Proof]

If we take h as the height of ΔABC at vertex C, then in the figure to the right

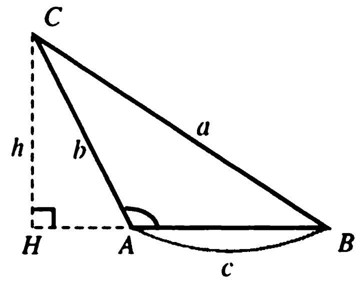
$$h = b \sin A.$$



Therefore,

$$S = \frac{1}{2}ch = \frac{1}{2}bc \sin A.$$

The other two parts of the formula can be derived analogously.



Problem

Find the area of the following triangles:

(1) $b = 2$ cm, $c = 5$ cm, $A = 60°$

(2) $a = 70$ m, $b = 40$ m, $C = 135°$

Topic for Enrichment: Heron's Formula

Heron's formula for finding the area of a triangle given the lengths of three sides is well known.

Let a, b, and c be the three sides of ΔABC, and let S be its area.

$$S = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where} \quad s = \frac{a+b+c}{2}.$$

Let's derive the above formula.

Starting from the formula

$$S = \frac{1}{2} bc \sin A,$$

if we multiply each side by two and then square each side, we obtain

$$\begin{aligned} 4S^2 &= b^2c^2\sin^2 A = b^2c^2(1-\cos^2 A) \\ &= b^2c^2(1+\cos A)(1-\cos A). \qquad (1) \end{aligned}$$

Since $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$,

$$\begin{aligned} 1+\cos A &= \frac{2bc+b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc} \\ &= \frac{(b+c+a)(b+c-a)}{2bc}. \end{aligned}$$

Analogously,

$$1-\cos A = \frac{(a+b-c)(a-b+c)}{2bc}.$$

If we let $a+b+c = 2s$, then

$$\begin{aligned} b+c-a &= 2(s-a) \\ a-b+c &= 2(s-b) \\ a+b-c &= 2(s-c). \end{aligned}$$

Therefore,

$$1+\cos A = \frac{2s(s-a)}{bc}, \qquad 1-\cos A = \frac{2(s-b)(s-c)}{bc}.$$

Substituting these expressions into (1), we get

$$S^2 = s(s-a)(s-b)(s-c).$$

Thus,

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

For example, let S cm^2 be the area of a triangle with sides of 4 cm, 5 cm, and 7 cm. Since $s = \dfrac{4+5+7}{2} = 8$,

$$\begin{aligned} S &= \sqrt{8\times(8-4)\times(8-5)\times(8-7)} \\ &= \sqrt{8\times4\times3\times1} = 4\sqrt{6}. \end{aligned}$$

Exercises

1. Find the length of BC in ΔABC, if $A = 120°$ and the radius of a circumscribed circle is 10 cm.

2. Let $BC = 14, AC = 10$, and $AB = 6$ in ΔABC. Find the measure of the largest of the three interior angles of the triangle.

3. What kind of triangle is ΔABC if the following equalities hold:

   (1) $a \sin A = b \sin B$ (2) $a \cos A = b \cos B$

4. Is a triangle with the following three sides an acute, right, or obtuse triangle?

   (1) 8, 10, 14 (2) 10, 24, 26 (3) 21, 25, 28

5. Using the table of trigonometric ratios, find the following values in ΔABC, if $A = 110°$, $b = 8$ cm, and $c = 13$ cm:

   (1) the length of side a (2) the area S

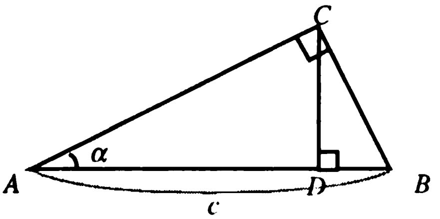
6. ΔABC is inscribed in a circle with a radius of 5 cm. Find the lengths of the three sides of ΔABC, if $A = 50°$ and $B = 60°$.

7. Prove that $S = \frac{abc}{4R}$, where S is the area of ΔABC and R is the radius of the circumscribed circle.

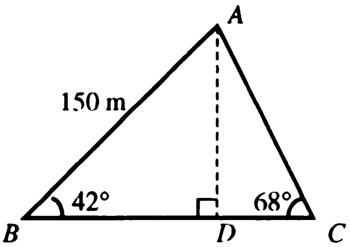
Chapter Exercises

A

1. Draw perpendicular CD from C to hypotenuse AB in right triangle ABC, where the right angle is $\angle C$. Express CD in terms of α and c, taking α as the measure of $\angle A$ and $AB = c$.



2. Find the lengths of AD and AC at the right using the table of trigonometric ratios.



3. Find the value of $\sin\theta$ if $\tan\theta = -2$. Assume that $0° \leq \theta \leq 180°$.

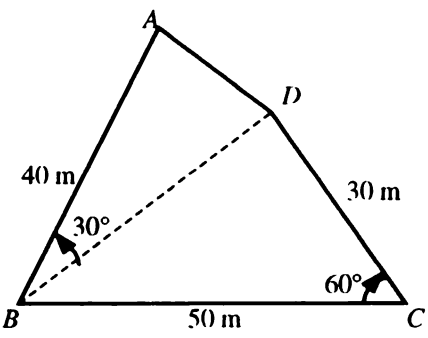
4. Find the following values if θ is obtuse and $\sin\theta = \dfrac{1}{2}$:

   (1) $\cos\theta$ (2) $\sin(180° - \theta)$ (3) $\tan(180° - \theta)$

5. We are given a quadrilateral-shaped plot of land $ABCD$, as in the figure to the right.

   (1) Find the length of BD.

   (2) Find the area of the plot.



6. What kind of triangle is ΔABC, if the equality $2\sin A\cos B = \sin C$ holds?

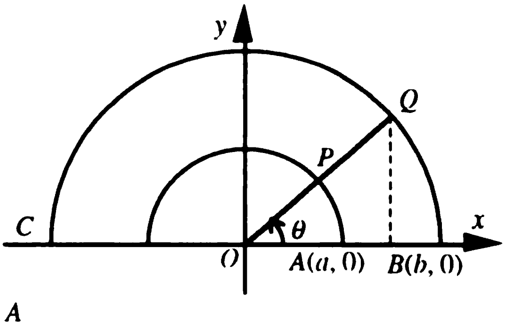
7. Prove that the following equalities hold for ΔABC.

   (1) $\sin(A + B) = \sin C$ (2) $\sin\dfrac{A}{2} = \cos\dfrac{B + C}{2}$

8. For ΔABC, $A : B : C = 3 : 4 : 5$ and the radius of a circumscribed circle is 2. Find the area of triangle ABC by dividing it into three triangles with the center of the circumscribed circle as their vertex and AB, BC, and CA as their bases.

B

1. Express the coordinates of points P, Q, and C in the figure to the right in terms of a, b, and θ.



2. What kind of triangle is ΔABC if the following equality holds?

$$(a - c \cos B) \sin B = (b - c \cos A) \sin A$$

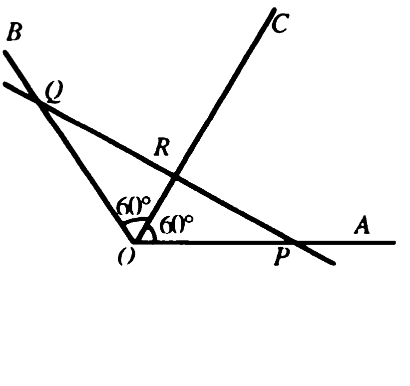
3. Find A, B, and C, if $a : b : c = \sqrt{2} : (1 + \sqrt{3}) : 2$ in ΔABC.

4. Prove that the area of a quadrilateral is $\frac{1}{2}ab \sin \theta$, with a, b as the lengths of its diagonals and θ as the angle formed by its diagonals.

5. Prove that the area S of a triangle can be expressed in terms of side a and angles B and C at the ends of that side as

$$S = \frac{a^2 \sin B \sin C}{2 \sin (B + C)} .$$

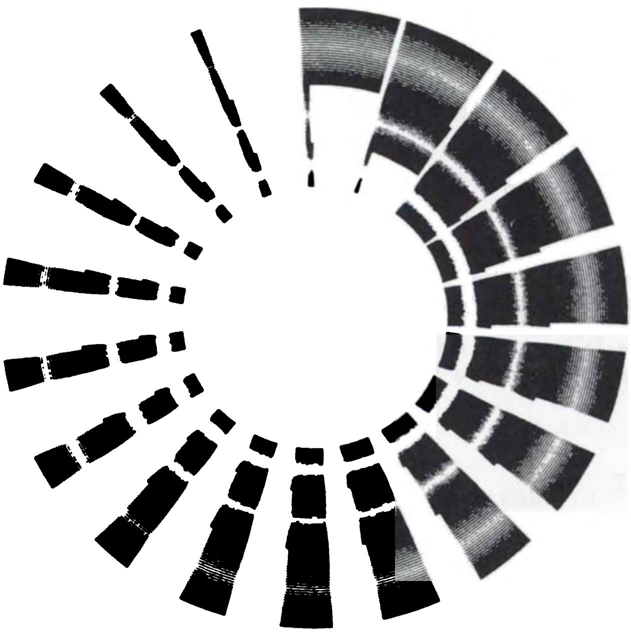
6. Let OC be the angle bisector of $\angle AOB$, which has a measure of 120°. Let P, R, and Q be the intersections of any line and OA, OC, and OB, respectively. We assume that $OP = p$, $OR = r$, and $OQ = q$. Prove that



$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

holds, using $\Delta OPR + \Delta OQR = \Delta OPQ$.

APPENDIX



Answers

Index

Greek Letters

Numerical Tables

B

1. 80 yen
2. $p < 2$
3. $\begin{cases} p = -1 \\ q = 1 \end{cases}$ $\begin{cases} p = -3 \\ q = 9 \end{cases}$
4. (1) $S = -6x^2 + 18x \quad (0 < x < 3)$

   (2) Maximum value $\frac{27}{2}$, $P(\frac{3}{2}, 3)$
5. $-2 < a < 6$
6. $y = \frac{2}{x+1} + 6 \quad (x > 0)$
7. $y = -\frac{10}{x+3} + 4 \quad (x > -3)$
8. For $k = -1$, 1

   for $k = -\frac{1}{2}$, 2

   for $k = 0$, 1

   for $k = \frac{1}{2}$, none.

Chapter 5 Trigonometric Ratios...................... 236-237

A

1. $CD = c \cos \alpha \sin \alpha$
2. $AD \approx 100.37$ m, $AC \approx 108.25$ m
3. $\sin \theta = \frac{2\sqrt{5}}{5}$, $\cos \theta = -\frac{\sqrt{5}}{5}$
4. (1) $-\frac{\sqrt{3}}{2}$ (2) $\frac{1}{2}$

   (3) $\frac{\sqrt{3}}{3}$
5. (1) $10\sqrt{19}$ m

   (2) $(375\sqrt{3} + 100\sqrt{19})$ m^2
6. An isosceles triangle with $BC = AC$.
8. $3 + \sqrt{3}$

B

1. $P(a \cos \theta, a \sin \theta)$, $Q(b, b \tan \theta)$, $C(-\frac{b}{\cos \theta}, 0)$
2. An isosceles triangle with $a = b$ or a right triangle with $\angle C$ as the right angle.
3. $A = 30°$, $B = 105°$, $C = 45°$

Index

Greek Letters

Capital	Small		Capital	Small	
Α	α	alpha	Ν	ν	nu
Β	β	beta	Ξ	ξ	xi
Γ	γ	gamma	Ο	o	omicron
Δ	δ	delta	Π	π	pi
Ε	ε	epsilon	Ρ	ρ	rho
Ζ	ζ	zeta	Σ	σ, ς	sigma
Η	η	eta	Τ	τ	tau
Θ	θ, ϑ	theta	Υ	υ	upsilon
Ι	ι	iota	Φ	φ, ϕ	phi
Κ	κ	kappa	Χ	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
Μ	μ	mu	Ω	ω	omega

Table of Squares, Square Roots, and Reciprocals

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$\frac{1}{n}$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$	$\frac{1}{n}$
1	1	1.0000	3.1623	1.0000	51	2601	7.1414	22.5832	0.0196
2	4	1.4142	4.4721	0.5000	52	2704	7.2111	22.8035	0.0192
3	9	1.7321	5.4772	0.3333	53	2809	7.2801	23.0217	0.0189
4	16	2.0000	6.3246	0.2500	54	2916	7.3485	23.2379	0.0185
5	25	2.2361	7.0711	0.2000	55	3025	7.4162	23.4521	0.0182
6	36	2.4495	7.7460	0.1667	56	3136	7.4833	23.6643	0.0179
7	49	2.6458	8.3666	0.1429	57	3249	7.5498	23.8747	0.0175
8	64	2.8284	8.9443	0.1250	58	3364	7.6158	24.0832	0.0172
9	81	3.0000	9.4868	0.1111	59	3481	7.6811	24.2899	0.0169
10	100	3.1623	10.0000	0.1000	60	3600	7.7460	24.4949	0.0167
11	121	3.3166	10.4881	0.0909	61	3721	7.8102	24.6982	0.0164
12	144	3.4641	10.9545	0.0833	62	3844	7.8740	24.8998	0.0161
13	169	3.6056	11.4018	0.0769	63	3969	7.9373	25.0998	0.0159
14	196	3.7417	11.8322	0.0714	64	4096	8.0000	25.2982	0.0156
15	225	3.8730	12.2474	0.0667	65	4225	8.0623	25.4951	0.0154
16	256	4.0000	12.6491	0.0625	66	4356	8.1240	25.6905	0.0152
17	289	4.1231	13.0384	0.0588	67	4489	8.1854	25.8844	0.0149
18	324	4.2426	13.4164	0.0556	68	4624	8.2462	26.0768	0.0147
19	361	4.3589	13.7840	0.0526	69	4761	8.3066	26.2679	0.0145
20	400	4.4721	14.1421	0.0500	70	4900	8.3666	26.4575	0.0143
21	441	4.5826	14.4914	0.0476	71	5041	8.4261	26.6458	0.0141
22	484	4.6904	14.8324	0.0455	72	5184	8.4853	26.8328	0.0139
23	529	4.7958	15.1658	0.0435	73	5329	8.5440	27.0185	0.0137
24	576	4.8990	15.4919	0.0417	74	5476	8.6023	27.2029	0.0135
25	625	5.0000	15.8114	0.0400	75	5625	8.6603	27.3861	0.0133
26	676	5.0990	16.1245	0.0385	76	5776	8.7178	27.5681	0.0132
27	729	5.1962	16.4317	0.0370	77	5929	8.7750	27.7489	0.0130
28	784	5.2915	16.7332	0.0357	78	6084	8.8318	27.9285	0.0128
29	841	5.3852	17.0294	0.0345	79	6241	8.8882	28.1069	0.0127
30	900	5.4772	17.3205	0.0333	80	6400	8.9443	28.2843	0.0125
31	961	5.5678	17.6068	0.0323	81	6561	9.0000	28.4605	0.0123
32	1024	5.6569	17.8885	0.0313	82	6724	9.0554	28.6356	0.0122
33	1089	5.7446	18.1659	0.0303	83	6889	9.1104	28.8097	0.0120
34	1156	5.8310	18.4391	0.0294	84	7056	9.1652	28.9828	0.0119
35	1225	5.9161	18.7083	0.0286	85	7225	9.2195	29.1548	0.0118
36	1296	6.0000	18.9737	0.0278	86	7396	9.2736	29.3258	0.0116
37	1369	6.0828	19.2354	0.0270	87	7569	9.3274	29.4958	0.0115
38	1444	6.1644	19.4936	0.0263	88	7744	9.3808	29.6648	0.0114
39	1521	6.2450	19.7484	0.0256	89	7921	9.4340	29.8329	0.0112
40	1600	6.3246	20.0000	0.0250	90	8100	9.4868	30.0000	0.0111
41	1681	6.4031	20.2485	0.0244	91	8281	9.5394	30.1662	0.0110
42	1764	6.4807	20.4939	0.0238	92	8464	9.5917	30.3315	0.0109
43	1849	6.5574	20.7364	0.0233	93	8649	9.6437	30.4959	0.0108
44	1936	6.6332	20.9762	0.0227	94	8836	9.6954	30.6594	0.0106
45	2025	6.7082	21.2132	0.0222	95	9025	9.7468	30.8221	0.0105
46	2116	6.7823	21.4476	0.0217	96	9216	9.7980	30.9839	0.0104
47	2209	6.8557	21.6795	0.0213	97	9409	9.8489	31.1448	0.0103
48	2304	6.9282	21.9089	0.0208	98	9604	9.8995	31.3050	0.0102
49	2401	7.0000	22.1359	0.0204	99	9801	9.9499	31.4643	0.0101
50	2500	7.0711	22.3607	0.0200	100	10000	10.0000	31.6228	0.0100

Table of Trigonometric Ratios

degrees	sin	cos	tan	degrees	sin	cos	tan
0°	0.0000	1.0000	0.0000	45°	0.7071	0.7071	1.0000
1°	0.0175	0.9998	0.0175	46°	0.7193	0.6947	1.0355
2°	0.0349	0.9994	0.0349	47°	0.7314	0.6820	1.0724
3°	0.0523	0.9986	0.0524	48°	0.7431	0.6691	1.1106
4°	0.0698	0.9976	0.0699	49°	0.7547	0.6561	1.1504
5°	0.0872	0.9962	0.0875	50°	0.7660	0.6428	1.1918
6°	0.1045	0.9945	0.1051	51°	0.7771	0.6293	1.2349
7°	0.1219	0.9925	0.1228	52°	0.7880	0.6157	1.2799
8°	0.1392	0.9903	0.1405	53°	0.7986	0.6018	1.3270
9°	0.1564	0.9877	0.1584	54°	0.8090	0.5878	1.3764
10°	0.1736	0.9848	0.1763	55°	0.8192	0.5736	1.4281
11°	0.1908	0.9816	0.1944	56°	0.8290	0.5592	1.4826
12°	0.2079	0.9781	0.2126	57°	0.8387	0.5446	1.5399
13°	0.2250	0.9744	0.2309	58°	0.8480	0.5299	1.6003
14°	0.2419	0.9703	0.2493	59°	0.8572	0.5150	1.6643
15°	0.2588	0.9659	0.2679	60°	0.8660	0.5000	1.7321
16°	0.2756	0.9613	0.2867	61°	0.8746	0.4848	1.8040
17°	0.2924	0.9563	0.3057	62°	0.8829	0.4695	1.8807
18°	0.3090	0.9511	0.3249	63°	0.8910	0.4540	1.9626
19°	0.3256	0.9455	0.3443	64°	0.8988	0.4384	2.0503
20°	0.3420	0.9397	0.3640	65°	0.9063	0.4226	2.1445
21°	0.3584	0.9336	0.3839	66°	0.9135	0.4067	2.2460
22°	0.3746	0.9272	0.4040	67°	0.9205	0.3907	2.3559
23°	0.3907	0.9205	0.4245	68°	0.9272	0.3746	2.4751
24°	0.4067	0.9135	0.4452	69°	0.9336	0.3584	2.6051
25°	0.4226	0.9063	0.4663	70°	0.9397	0.3420	2.7475
26°	0.4384	0.8988	0.4877	71°	0.9455	0.3256	2.9042
27°	0.4540	0.8910	0.5095	72°	0.9511	0.3090	3.0777
28°	0.4695	0.8829	0.5317	73°	0.9563	0.2924	3.2709
29°	0.4848	0.8746	0.5543	74°	0.9613	0.2756	3.4874
30°	0.5000	0.8660	0.5774	75°	0.9659	0.2588	3.7321
31°	0.5150	0.8572	0.6009	76°	0.9703	0.2419	4.0108
32°	0.5299	0.8480	0.6249	77°	0.9744	0.2250	4.3315
33°	0.5446	0.8387	0.6494	78°	0.9781	0.2079	4.7046
34°	0.5592	0.8290	0.6745	79°	0.9816	0.1908	5.1446
35°	0.5736	0.8192	0.7002	80°	0.9848	0.1736	5.6713
36°	0.5878	0.8090	0.7265	81°	0.9877	0.1564	6.3138
37°	0.6018	0.7986	0.7536	82°	0.9903	0.1392	7.1154
38°	0.6157	0.7880	0.7813	83°	0.9925	0.1219	8.1443
39°	0.6293	0.7771	0.8098	84°	0.9945	0.1045	9.5144
40°	0.6428	0.7660	0.8391	85°	0.9962	0.0872	11.4301
41°	0.6561	0.7547	0.8693	86°	0.9976	0.0698	14.3007
42°	0.6691	0.7431	0.9004	87°	0.9986	0.0523	19.0811
43°	0.6820	0.7314	0.9325	88°	0.9994	0.0349	28.6363
44°	0.6947	0.7193	0.9657	89°	0.9998	0.0175	57.2900
45°	0.7071	0.7071	1.0000	90°	1.0000	0.0000	∞

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