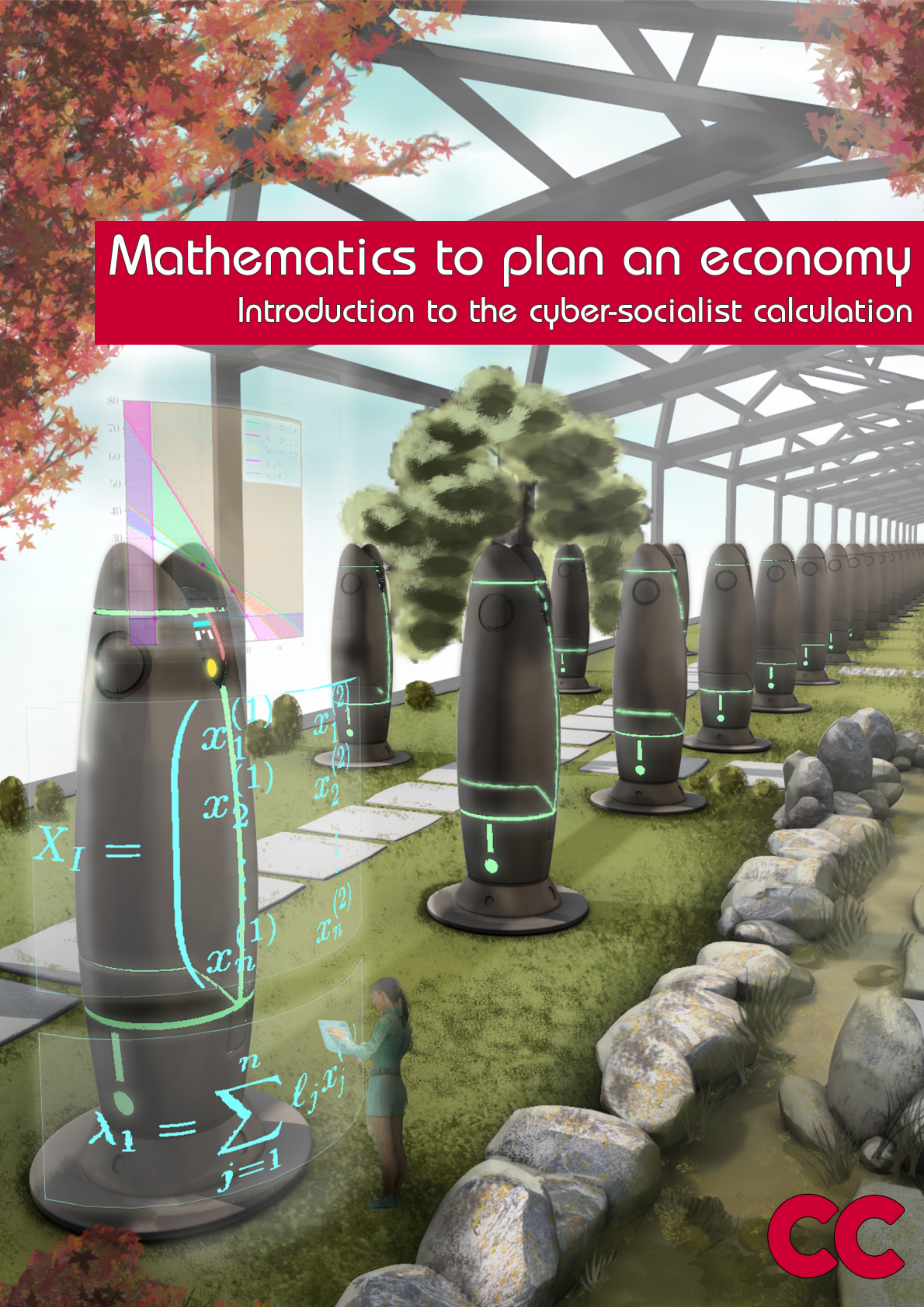
Mathematics to plan an economy

Introduction to the cyber-socialist calculation



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Índice general

The purpose of this guide is to help all those interested in understanding the more technical aspects of the emerging cybernetic communism. So far, there have been some introductory and some very advanced writings[1] on this topic, with a conspicuous lack of ”bridges” between one and the other. At CibCom, we aspire to transcend this situation; we want to guide novices who venture into the (obscure at first sight) mathematical and computer science fundamentals of economic planning without ever having specialized in it or not having worked on it for some time. The general public may consult the document when they encounter difficulties in interpreting an algebraic idea or expression in any of these texts. It is especially extensive because we have not wanted to skip any of the explanations that are usually taken for granted in the treatises on this subject.

In any case, before entering into what brings us here today, it is convenient to remember the context that, for almost three centuries, has been the reason for these investigations.

1 Introduction

The socialist planning of the economy – the flagship of cyber-communism – is one of the ways of organizing and coordinating production in modern societies, characterized by a highly developed technical division of labor. In the face of this, the prevailing market economy rises antagonistically. It produces and reproduces in the world a historically singular reality.

In capitalist society, characterized by the private ownership of the means of production, conscious coordination is non-existent and organization occurs at the atomic level (in companies). Capitalist planning, in spite of how much it has been technified in the last decades, occurs only within individual companies and, more importantly, it is fundamentally oriented towards profit expectations. Between different private companies, it is no longer that there is no harmonious planning, it is that there is no planning at all.[2] Only *a posteriori*, and according to the logic of blind and impersonal market automatism, can the different productive units be coordinated to supply the demands

[1] See the very rich literature that the original works of Otto Neurath, Wassily Leontief, Leonid Kantoróvich, Oskar Lange, Víktor Glushkov, Nikolay Veduta, Stafford Beer, Paul Cockshott and Allin Cottrell, Jan Philipp Dapprich, Spyridon Samothrakis, Tomas Härdin, etc., represent.

[2] The staunchest defenders of capitalism, the Austrian economists, displaying a palpable straw man, directly deny the very *possibility* of a conscious coordination at the social level. From their parameters, planning (or solving the allocation problem in a socialized economy) would necessarily imply leaving the role of making all economic decisions in the hands of a single authority. To do so, this authority would have to be omniscient; that is, it would have to have access to the most detailed economic information and know the best use for each and every one of the millions of resources in the economy. Given

of the people. These demands (which are sometimes the most basic human needs) will be satisfied, or not, exclusively according to the level of income of each person and the availability of goods that each country has in the global supply chain.

	Captialist Market	**Socialist Planning**
Coordination	Automatic, via competition	Conscious, through political institutions
Economic calculation	Monetary	In kind
Target	Private profit	Social needs
Climate constraints	Introduced from outside disrupting the operation (negative externalities)	Organically applied in the planning process itself
Employment	According to the needs of capital (structural unemployment)	According to individual will and social needs
Political form	Parliamentarism / Bureaucratic State in charge of ensuring capital accumulation / Military dictatorship	Direct Democracy / Commune

Cuadro 1: Comparison table between systems.

What the apologists of capitalism sell us as a rational organization of production is revealed to be radically problematic, with inherently disastrous dynamics for humanity and the environment. Many of the features of our economic reality, such as extreme income inequality and recessions, are necessary consequences of the social relations of production and thus enduring and essential properties of capitalism, rather than accidental or transitory. Despite the prognostications of market advocates, all this continues to occur, along with the immense human suffering they entail.

These dynamics are the source of innumerable political conflicts. Social polarization becomes more acute during crises, constantly permeating our coexistence. The inability to exercise social control over atomized economic activity is reflected in the absolute prostration of the ”liberal democracies” before the designs and needs of ca-

the obvious impossibility of such authority, the Austrians would propose as the only efficient solution to the allocation problem the decentralized mechanism of the market [1].

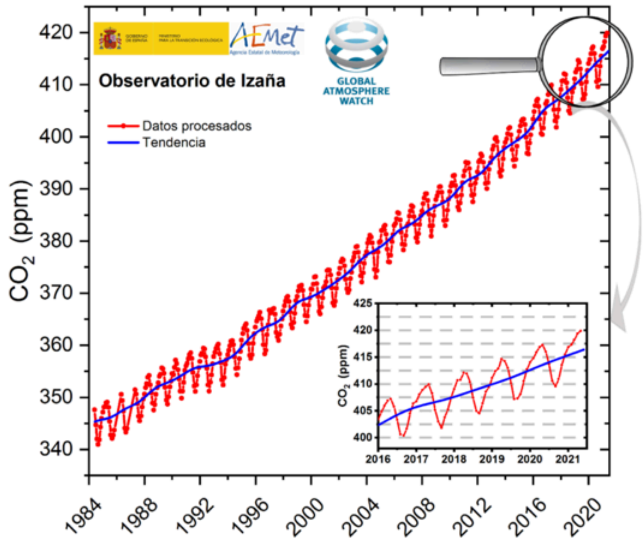

Figura 1: Monthly averages of CO_2 concentration (in ppm) (red dots) along with the trend of CO_2 (blue line)[3].

pital. All these are sufficient reasons to investigate the possibility of a conscious and democratic organizational alternative. However, preventing capitalism from undermining the planet's capacity to sustain life sticks out as the greatest challenge of all due to its urgency.

Our planet has dynamics that allow the renewal of resources essential to sustain any form of life, but the current production of goods constantly exceeds its biophysical limits. In the long run this system constitutes a danger to the prolongation of human life. Climate change and resource limitation are just beginning to manifest themselves. Our current situation is only the prelude to what is to come: first crises of scarcity, major difficulties for agriculture in several areas of the planet, increasing emissions of greenhouse gases, aridification of entire regions, flooding of others, etc [2]... In addition, many resources are wasted every year due to the large portion of goods that cannot be exchanged. This is going to be the greatest global challenge humanity has ever faced in its short but frenetic history. It is a challenge the market has generated itself but is unable to address, because the imperious need for growth makes capitalism incapable of facing it. In the light of this new danger, its behavior is even more irrational and harmful.

Despite the evidence, economists, politicians, intellectuals, social democrats, liberals and other political tendencies have been trying for years to find solutions that somehow magically eliminate the negative effects of the market without changing what they consider the greatest success of capitalism, namely accumulation. In other words, they propose to fix the issues of the market without altering its nature. In doing so, they do not need to think about or accept the difficulties of transforming the economic basis of society. The momentary advances of the workers' movement are in full retreat with the disarticulation of the so-called "welfare state". Unemployment persists and social crises are the order of the day. Pollution continues to worsen and the energy transition is not progressing as planned. Capitalist accumulation suffers repeated constraints. Discontent encourages political forces that erode the institutions fundamental to sustaining the reproduction of capital.

We propose to address the root of the problem: we propose the alternative of a democratic, conscious and rational planning of the economy, to enable us to face these and other challenges, but also to freely choose our destiny as a society. Could this be a utopian chimera such as that which the great capitalist media sells us on a daily basis? Has this not been tried before, with disastrous results? Planning is not an entirely new, 21st century concept. In fact, the labor movement has been considering the option of a planned economy as an alternative to capitalist disorder since the 19th century. These ideas can be traced through the different organizational and political expressions of our class, especially in the Internationals I, II and III, and it was the triumph of the October Revolution in 1917, which allowed the most ambitious approach to economic planning in history to be carried out in the Soviet Union.

The Soviet economy was in an extremely precarious situation as a result of a bloody and destructive 6-year civil war. After a short period of New Economic Policy with capitalist elements, the context of international isolation and the need for rapid industrial development led the Soviets to try to rationally organize a national economy focused on self-sufficiency. This challenge was immense: it was an enormous territorial extension where capitalism had only been partially implemented, with social relations more typical of feudalism [4]. In the first five-year plan of 1928, the aim was to transform a predominantly agrarian economy into one with a strong base of heavy industry. At that stage, Soviet society experienced a rapid growth in its wealth: "Soviet national income at constant 1928 prices grew by more than 60 percent" between 1928 and 1933 [4]. Economic planning succeeded in establishing the USSR as a world power despite its obvious initial backwardness and having to simultaneously repel the Nazis in what was one of the most destructive invasions in human history. However, after the significant achievements of that first planning phase, problems would soon appear in the organization of an increasingly diverse and complex economy, such as shortages and gaps in production chains, which the USSR would face until its dismantling in the 1990s. These were partly due to the deficient planning methods adopted at the time, owing to a the lack of computer capacity. As a result of the inability to efficiently process economic information by means of electronic calculations, the Soviet system suffered from the following three problems:

1. As the planning body could not calculate how much direct and indirect labour it would cost to produce each good, the prices of these goods ended up being fixed on the basis of subjective criteria ("most basic", cheapest, "most superfluous", most expensive, etc.) [5, 6]. Thus, goods that were relatively difficult to produce were sold at prices well below their cost, leading to shortages on the consumer side and mismatches in state accounting. This point is discussed in more detail in the Appendix A.

2. Planning was not based on society's actual demand, but rather on raw production targets (tonnes of coal, iron etc.). In other words, the final goods consumed by the population were not considered to determine the required amounts of raw materials, but a base quantity of raw materials was estimated instead, which was then gradually transformed until the final good finally reached, whenever it was possible, the final consumer. This led to a huge waste of human and material resources that ultimately weighed down the Soviets.

3. As a consequence of the above, the use of money as a unit of calculation and method of payment was retained. This was problematic, not only because, as shown by [7], [8] and [9], it spontaneously engenders high inequalities and, therefore, political conflicts incompatible with the real democracy we long for, but because of a deeper question. What information do monetary expressions or market prices provide? At most, the relative amount of labor socially necessary to produce a good and its relative scarcity [10]. Questions such as how polluting the production of a good is, how long it takes for ecosystems to regenerate a raw material or the workers' suffering it causes, are categorised as "negative externalities", incurring a systematic loss of information that harms the effective operation of the system.

The inefficiencies resulting from the lack of non-computerized economic planning paved the way to the progressive restoration of mercantile dynamics in the USSR. The multiple achievements of Soviet socialism were gradually dismantled, a process that began with the "Kosygin reform" and concluded with the Perestroika, thereby leading to the destruction of the USSR and the subsequent establishment of nationalist and neoliberal regimes that today dominate the ex-Soviet republics.

In the current day and age, however, all the aforementioned problems can quite easily be solved. The study of such problems, together with technological innovations, have made an unprecedented theorization of economic planning possible, with direct democracy and *calculation in kind* emerging as fundamental pillars. Instead of reducing economic rationality to a one-dimensional variable such as monetary profit, economic accounting in physical units, integrated into some sort of public plebiscite, makes it possible to organically use multidimensional criteria such as scientific recommendations (ecology, public health, etc.) and ethico-political values (labour dignity, intergenerational justice, international solidarity, etc.) [11, 12]. With that in mind, it is the aim of this document to act as an introduction to the formal expressions of new economic coordination techniques.

In particular this article addresses three main topics: matrix computation, optimization and computational complexity. These three topics aim to respectively solve the problems of *logistics* (how to ensure that we produce neither more nor less than what is needed), *development* (how to update our production in relation to changing social and technological situations) and *feasibility* (how to be sure that computations are performed in a reasonable time and with sufficient approximation).

It should be clarified that, by sticking to the "technical" scope of planning (i.e., mathematics and computation), we are intentionally omitting all the equally necessary legal-political requirements for the conscious and democratic organization of the production of goods and services. This is a broad topic on its own that we plan to address in future articles.

Let's plan!

2 Matrix Theory

The reader has probably wondered at some point which manufacturing processes a good, for example a cell phone, has to go through until it reaches our home in a functional state. As you can imagine, it is a tremendously complex process; its manufacture includes everything from energy generation and mineral extraction to the manufacture of the semiconductors and plastics used, among many other things.

The market does not directly coordinate all these processes as a whole, however, despite its atomicity, it is capable of linking different production units around the globe to satisfy an effective demand. The basic mechanisms by which its social metabolism is regulated are: 1) the discipline exercised by competition between companies and 2) the price system and cash flow. These and the feedback they generate result in a network of signals and information flows that are capable of indirectly incorporating social material costs into the monetary cost of goods.

Our proposal aims to overcome these mechanisms and to be as or more effective in organizing the productive process. Let us place ourselves for a moment in the place of a committee appointed to coordinate the economy as a whole, within the parameters that the citizens have decided. How do these public officials correctly organize the distribution and production of resources? How can they solve *logistical problems of supply and inventory* in such a way as not to generate shortages, in the absence of competition? For this, it is necessary to be able to understand the relationships between the different production units in a scientific way.

Consequently, at the very least, it must be possible to compute the social costs of production of each type of good directly and precisely, not as a result of laws beyond the conscious control of human beings. To achieve this and capture the social metabolism, it is now possible to use the so-called "technological matrix". Thanks to this mathematical object, we will be able to know, in a much more rigorous way than in the Soviet era, the adequate valuation for each good, as well as the amount of resources needed to produce it. But first, in order to define it, we must explain the concept of the matrix and its properties.

2.1 Basic Operations or: "What in the world is a matrix and how do we deal with it?"

A matrix is nothing more than an ordered, two-dimensional array of numbers, that is, a set of numbers arranged in rows and columns. Formally, a matrix A with n rows and m columns is defined as

$$A := \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,1} & a_{2,3} & \cdots & a_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,m} \end{bmatrix},$$

or in abbreviated form $A = (a_{i,j})_{i=1,\dots n, j=1,\dots m}$. Note that $a_{i,j}$ is simply the number occupying the i-th row and the j-th column.

In order to know the position of any element in a table, you only need to look up its row and column. In itself, a matrix does not mean anything, but that does not mean that we cannot give it a meaning. A matrix with a single row and a single column can be understood as just an ordinary number. On the other hand, to define a geographical point on the globe, a single number is not enough, we need at least two (latitude and longitude), which could be represented by a matrix with one row and two columns (or with two rows and one column). Another example of using matrices, but with more rows and columns than before, would be: the information from a catalog with 20 car models classified by 5 different characteristics (e.g., weight, height, engine power, number of doors and price); can be gathered in a matrix with 20 rows and 5 columns or with 5 rows and 20 columns, depending on the reader's choice, there is no mathematical reason to choose the former over the latter. However, the former is more common for a catalog.

A very special kind of matrices widely used in our day-to-day life are matrices with a single row and two columns, because, as mentioned before, they can represent points on a plane. For example, $\mathbf{p} = [1, 2]$ would be, set to a unit of distance, the point on the plane whose latitude is 1 and whose longitude is 2. Thus the points could be represented on a map. In general, row matrices are matrices with a single row and column matrices are matrices with a single column, these types of matrices are refered as *row* and *column vectors* respectively. Another kind of peculiar matrices are those which have the same number of rows and columns, these are called square matrices, and have specially important mathematical properties.

Now that we know what is a matrix, lets see what operations can they be involved in.

2.1.1 Sum of matrices

The first operation one can think of is adding matrices. How can we define this operation? The following definition of matrix addition requires that both matrices involved have exactly the same number of rows and columns. Formally, let A and B be matrices of the form

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,m} \end{bmatrix}, \quad B = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,m} \\ b_{2,1} & b_{2,2} & b_{2,3} & \cdots & b_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & b_{n,3} & \cdots & b_{n,m} \end{bmatrix},$$

then we define the sum matrix of A and B as

$$A + B := \begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & a_{1,3} + b_{1,3} & \cdots . & a_{1,m} + b_{1,m} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & a_{2,3} + b_{2,3} & \cdots & a_{2,m} + b_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} + b_{n,1} & a_{n,2} + b_{n,2} & a_{n,3} + b_{n,3} & \cdots & a_{n,m} + b_{n,m} \end{bmatrix};$$

i.e. the sum matrix is the matrix whose element in the i-th row and j-th column is the sum of the respective elements of A and B occupying that position. For this definition to work, the matrices have to be the exact same size, since, the sum of matrices of different size does not yield meaningful results (What would summing a vector with a matrix even mean?). Summing two (or more) matrices is simply writing a matrix whose elements are the "cell" to "cell"[3] sums of the matrices involved. Note that $A+B = B + A$, i.e., matrix addition is a commutative operation. In an analogous way we can define subtraction of matrices:

$$A - B := \begin{bmatrix} a_{1,1} - b_{1,1} & a_{1,2} - b_{1,2} & a_{1,3} - b_{1,3} & \dots & a_{1,m} - b_{1,m} \\ a_{2,1} - b_{2,1} & a_{2,2} - b_{2,2} & a_{2,3} - b_{2,3} & \dots & a_{2,m} - b_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} - b_{n,1} & a_{n,2} - b_{n,2} & a_{n,3} - b_{n,3} & \dots & a_{n,m} - b_{n,m} \end{bmatrix}.$$

A concrete example of this operation would be

$$\begin{bmatrix} 1 & -4 & 3 \\ 3.1 & 9 & 5 \end{bmatrix} + \begin{bmatrix} 0.5 & 4 & 3 \\ 0.1 & -11 & 30 \end{bmatrix} = \begin{bmatrix} 1.5 & 0 & 6 \\ 3.2 & -2 & 35 \end{bmatrix}.$$

[3]Sometimes, a matrix sometimes is referred to as a *table* which contain each item inside of a *cell*. This specialized is specially common in fields related to computer science.

2.1.2 Matrix scalar multiplication

Let w be any real number, then we define the scalar multiplication of a matrix A by w as:

$$wA = Aw := \begin{bmatrix} wa_{1,1} & wa_{1,2} & wa_{1,3} & \cdots & wa_{1,m} \\ wa_{2,1} & wa_{2,2} & wa_{2,3} & \cdots & wa_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ wa_{n,1} & wa_{n,2} & wa_{n,3} & \cdots & wa_{n,m} \end{bmatrix};$$

i.e. we multiply the contents of each cell by the scalar w. One could say that, when the matrix is multiplied by a scalar, which is no more than a constant number (e.g., $w = 4$), we are literally "scaling" the whole matrix by that number, hence the name. Note that since the product of two real numbers is commutative, the scalar multiplication of a matrix also is.

2.1.3 Matrix multiplication

The next operation between matrices that one can think of is the multiplication or product of matrices. Matrix addition and scalar multiplication were defined in a "cell"-by-"cell" fashion, however, matrix multiplication is NOT.[4] The reason as to why the standard definition of matrix multiplication doesn't follow this rule may not be obvious at first, however, it is the most useful definition due to its links with linear equations and mappings.

Remember that we are defining operations between matrices, guaranteeing that these are well defined. Everything is valid as a definition, but, is the usefulness of those objects and operations defined whats important to mathematicians. For a mathematician, it makes no sense to question the definitions themselves, they are not proved, only their consistency and properties are. Before defining the product of matrices in general, let us define it for particular cases. Let us define the product of a matrix A with n rows and m columns by a column matrix (column vector) x with m rows;

[4] Although, there is a matrix multiplication definition that works this way called the Hamadar product which is useful in the JPEG algorithm.

$$Ax = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ a_{2,1} & \cdots & a_{2,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} := \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,m}x_m \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,m}x_m \\ \vdots \\ a_{j,1}x_1 + a_{j,2}x_2 + a_{j,3}x_3 + \cdots + a_{j,m}x_m \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + a_{n,3}x_3 + \cdots + a_{n,m}x_m \end{bmatrix} .$$

Do not panic! The method is very simple; 1) take the first row of the matrix and multiply it by the element-by-element vector and add the result 2) place the result in the first row of the resulting column vector 3) repeat the operation with the following rows of the matrix (but placing the result of 1) in the corresponding row of the resulting column vector).

A concrete example of matrix-vector product would be

$$\begin{bmatrix} 0.1 & 0.40 \\ 0.50 & 0.25 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.1 \cdot 2 + 0.40 \cdot 3 \\ 0.50 \cdot 2 + 0.25 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 1.75 \end{bmatrix} .$$

From a matrix with n rows and m columns and a column vector (with exactly m rows) we obtain a column vector with n rows.

We proceed to define matrix multiplication in general, of which matrix-vector multiplication is a particular case. Let A be a matrix with n rows and m columns and B be a matrix with m rows and p columns. If we write

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,m} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & b_{2,3} & \cdots & b_{2,p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & b_{m,3} & \cdots & b_{m,p} \end{bmatrix} ,$$

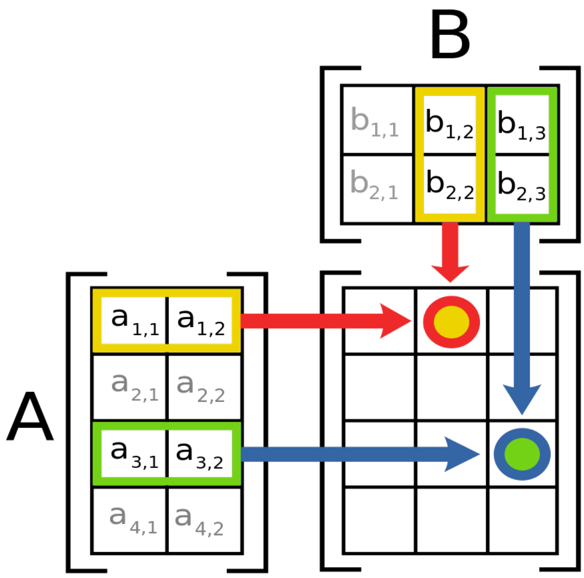

Figura 2: Schematic depiction of the matrix product AB of two matrices A and B.

then we define the multiplication of A by B as the matrix of n rows and p columns following

$$AB := \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,p} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,p} \end{bmatrix},$$

where $c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + a_{i,3}b_{3,j} + \ldots + a_{i,m}b_{m,j}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots., p$, i.e., the element occupying the i-th row and j-th column of the product matrix is the sum of the element-by-element multiplications of the i-th row of A by the j-th row of B. We note that 1) for this definition to make sense the number of columns that A must have and the number of rows that B must have must coincide and 2) the resulting matrix has the same rows as A and the same number of columns as B. Consequently, if the number of rows of A does not coincide with the number of columns of B then BA is not well defined, so the commutative property is not true in general. However, one might ask in case both A and B had n rows and n columns, could $AB = BA$? Contrary to the sum of matrices, the answer is negative, in general matrix multiplication does not commute. Let's look at a couple of simple examples of this phenomenon. In the first one we have

$$\begin{bmatrix} 2 & -3 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 0 & 10 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 0 + (-3) \cdot 3 & 2 \cdot 10 + (-3) \cdot (-1) \\ 7 \cdot 0 + 5 \cdot 3 & 7 \cdot 10 + 5 \cdot (-1) \end{bmatrix} = \begin{bmatrix} -9 & 23 \\ 15 & 65 \end{bmatrix}$$

while

$$\begin{bmatrix} 0 & 10 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 0 \cdot 2 + 10 \cdot 7 & 0 \cdot (-3) + 10 \cdot 5 \\ 3 \cdot 2 + (-1) \cdot 7 & 3 \cdot (-3) + (-1) \cdot 5 \end{bmatrix} = \begin{bmatrix} 70 & 50 \\ -1 & -14 \end{bmatrix} .$$

Or another example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ;$$

meanwhile

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 1 \cdot 0 & 1 \cdot 1 + 1 \cdot 1 \\ 0 \cdot 0 + 0 \cdot 0 & 0 \cdot 1 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} .$$

2.1.4 Identity matrix and inverse matrix

We define the n-dimensional identity matrix

$$I_n := \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{n \text{ columns}}$$

as the square matrix whose diagonal elements (those occupying the same row as column, i.e., the elements of the form $a_{i,i}$) are ones and all other entries are null. Note that an identity matrix I[5] verifies that $I\mathbf{x} = \mathbf{x}$ for every column vector $\mathbf{x}$, hence its name.

We will say that C is the inverse matrix of A if it verifies

$$AC = CA = I . \tag{1}$$

Given a matrix A there is, at most, only one matrix that satisfies the former equality. We call the matrix A^{-1} that verifies

$$AA^{-1} = A^{-1}A = I$$

[5] By I we refer to any identity matrix with an arbitrary number of columns.

the inverse matrix of A. Indeed, if C_1 and C_2 satisfy (2.1.4) then

$$C_1 = C_1 I = C_1(AC_2) = (C_1 A)C_2 = IC_2 = C_2$$

where we have taken into account that $F(GH) = (FG)H$ for any F, G, H square matrices.[6] Also, it can be proved that if A^{-1} exists then A is a square matrix.[7] Additionally, if A^{-1} and B^{-1} both exist, then $(AB)^{-1} = B^{-1}A^{-1}$.

2.1.5 Transpose

Given a matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,m} \end{bmatrix},$$

we define the *transpose* of the matrix A as

$$A^T = \begin{bmatrix} a_{1,1} & a_{2,1} & a_{3,1} & \dots & a_{m,1} \\ a_{1,2} & a_{2,2} & a_{3,2} & \dots & a_{m,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & a_{3,n} & \dots & a_{m,n} \end{bmatrix};$$

i.e. A^T is formed by turning rows of the matrix A into columns and vice versa. For example, if

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

then

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix};$$

or if

$$\mathbf{x} = [x_1, \dots, x_m],$$

[6] This property is called associativity and should be proven. The arithmetic proof is simple but cumbersome. Alternatively, you could prove this fact by taking advantage of the fact that a matrix represents a linear function, and that multiplication as defined before represents the composition of said linear functions. Since the composition of functions trivially satisfies the associativity, then so does matrix multiplication [13, Chapter 9].

[7] In [14] you can consult an elementary algorithm to compute the inverse matrix of A (if it exists) using the Gaussian elimination method used to solve systems of linear equations can be found in the Section "Complexity" and its computational complexity is explained and checked.

then

$$\mathbf{x}^T = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} .$$

2.1.6 Relation between solutions of systems of linear equations and inverse matrices

Before concluding this section let's look at the relationship that exists between invertible matrices and systems of linear equations with a unique solution.[8] Starting with a particularly simple example, let us consider the following system of linear equations:

$$\begin{cases} x + y = 1 \\ x - y = 1 \end{cases} \tag{2}$$

How could we express these relations in a matrix form? This is where the definition of matrix multiplication comes in handy, we can write (2) as

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Letting $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, i.e. the matrix whose rows are the coefficients of the system of linear equations and $\mathbf{c} = [1, 1]^T$ it turns out that (2) can be represented as

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{c} .$$

If we solve (2) by adding the first and second equations, we obtain that $2x = 2$, therefore, $x = 1$ and $y = 0$ is our solution.

There is another interesting way to solve this problem. First, by applying the method described in [14] or in section 4.2 we obtain that $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$. It turns out that $\mathbf{c}$, we get $A^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is the previously obtained solution.[9] This is no coincidence. Let us consider the general case of the phenomenon that has just occurred

[8] Also known as consistent independent system.

[9] The more skeptical reader can verify that $AA^{-1} = A^{-1}A = I$.

in order to shed some light on it. Consider the following system of n linear equations with unknowns $x_1, \ldots, x_m$:

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = y_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m = y_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m = y_n \end{cases} \tag{3}$$

with $a_{i,j}$, y_i given for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. It turns out that (3) is equivalent to solving the problem of finding the column vector $\mathbf{x}$ such that

$$A\mathbf{x} = \mathbf{y}\,, \tag{4}$$

with $A = (a_{i,j})_{i=1,\ldots,n,\, j=1,\ldots,m}$ and $\mathbf{y} = [y_1, \ldots, y_m]^T$. Notice that if A^{-1} exists, it follows by multiplying from the left on both sides of (4) that $A^{-1}A\mathbf{x} = I\mathbf{x} = \mathbf{x} = A^{-1}\mathbf{y}$ and thus, $\mathbf{x} = A^{-1}\mathbf{y}$ is the solution of (3).

This method of solving systems may feel too complicated or cumbersome to use over the standard, manual symbolic method at first; but it is interesting to note that if you changed the values of $y_1, \ldots, y_n$ in (3), the new problem does not have to be solved from scratch again, since one can take advantage of the fact that A^{-1} is already computed and would only need to calculate a single product. So if you had to solve multiple systems with the same coefficients, you would probably save time by using the second method.

Another reason for using the second method is that it obtains the solutions (or lack of thereof) in a finite number of basic arithmetic operations between the coefficients $a_{i,j}$ and the constant terms $y_1, \ldots, y_n$ of any system of linear equations. Computers are not capable of solving equations in symbolic mode like we do, but they are capable of storing matrices and vectors and performing operations on them *very* efficiently.[10]

Now that we understand some matrix operations, we are prepared to take a peek into some of the important applications that matrices (and linear algebra as a whole) have in the field of economic science and, in particular, in economic planning.

[10] See BLAS.

2.2 Why is linear algebra important to our proposal?

As previously discussed, the system employed by the USSR had at least two major and unavoidable flaws. The inability to respond to these problems led to a discrediting of the planned economy that served largely as a bonus for those who advocated the incorporation of capitalist-oriented reforms, which only aggravated the problems and worsened the quality of people's lives. We are going to try to illustrate how modern mathematics, in particular matrix theory, provide an answer to both flaws described and how they serve as a foundation for a planned economy. In this section our aim is not to deal with socialist planning in all its complexity, but only to illustrate the basic mathematics that form the foundation for further research.

We will work under certain assumptions, which we will discuss below, in order to simplify the mathematical treatment. We are aware that a real economy requires taking into account many more variables, nevertheless, we will give some references and observations on the treatment of more nuanced cases.

The method below is inspired by, and generalizes the analyses carried out by W. Leontief, for which he was awarded the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel (Swedish: *Sveriges riksbanks pris i ekonomisk vetenskap till Alfred Nobels minne*) in 1973. His method of analysis of the economy, itself inspired by François Quesnay, Leon Walras, Karl Marx and the Soviet planning, is called the "input-output method", which succinctly consists of using matrix algebra to describe the intersectoral relationships of an economy in general equilibrium.[11] [12] Any reader interested in original sources in which Leontief describes and uses his powerful method may consult [15], [16] and [17].

Suppose a closed economy (i.e., no flows with any external agent) with m different types of items, the first n types of items being production goods, used to produce other goods (wood, steel, industrial machinery,...etc) and the remaining $m-n$ types of items being consumption goods, those that are not used to produce other goods (chocolate milkshake, medicines, bandages,...etc).[13] Let's assume that:

(a) Each sector produces only one type of good, therefore there are no intrasector intermediate goods.[14]

[11] The notion of general equilibrium means, informally, that supply and demand of all types of products coincide.

[12] It is interesting to note that W. Leontief worked with market prices, not with magnitudes that are independent of the sphere of circulation, as we will do below.

[13] Consumpition goods are those that are not used to produce other goods.

[14] This assumption is not difficult to remedy, e.g., the reader may refer to [18].

(b) There is no training or education that workers can acquire, either to do certain jobs that they would not otherwise be able to do or to maximize their productivity.[15]

(c) All production assets have the same life span, which is taken as unitary. The wear and tear of heavy machinery, factory infrastructure ... etc. will not be taken into account.[16]

(d) All types of goods require the same manufacturing time, which is taken as a unit of time.[17]

(e) There are no different production techniques available for each type of good.

While every other assumption has some workarounds, the last assumption is especially unrealistic and problematic, you would only have to watch Breaking Bad to know that you can produce any specific good in different ways. The different techniques could be chosen based on efficiency by comparing their *integrated labor cost* (defined below)[18] and by the final amount of a good that has to be produced (represented by the final demand) In any case, this crucial question is beyond the scope of this article. The reader interested in how to deal with this phenomenon can consult [22], [23], [24] where the issue is succinctly discussed. There are several authors, such as Tomas Härdin and David Zachariah, who are studying these issues in depth at the present moment.

Let $a_{j,i}$ be the *amount of units* of the type of good j [19] required to produce a unit of the type of good i [20] and ℓ_i the *amount of direct labor hours required* to assemble one unit of the type of good i. Thus, the production of good type i is characterized by the vector $[a_{1,i}, ..., a_{n,i}, \ell_i]$, which represents the inputs required and the work of assembling those inputs to produce, after a certain (production) period, one unit of good type i.The amount of labor that a unit of the good of type i embodies is the sum of the direct labor employed in its production and the labor "“stored” in the inputs used. We

[15] The reader interested in addressing this aspect can consult [19, Chapter 2].

[16] The interested reader can read [20] for more information in this respect. In [19] the (uneven) wear and tear of machinery is taken into account, it does not pose a great difficulty.

[17] See [21], where it is briefly discussed. Further analysis on this point will be necessary as it is an important variable beyond "integrated labor costs" and environmental considerations, which has hardly been addressed.

[18] A perceptive reader may say that a certain technique can be more efficient than another if the chosen set of techniques to produce the other types of goods is different, and they would be right.

[19] The units of $a_{j,i}$ are physical units fixed in advance. For example, if the type of good j was bread, we would use the kilo or gram as the unit of reference.

[20] Once the economy is disaggregated by sectors, $a_{j,i} = \frac{z_{j,i}}{x_i}$ where $z_{j,i}$ is the quantity of goods produced in sector j that are inputs to sector i (in a capitalist economy the unit of $z_{j,i}$ and the rest of variables can simply be monetary) and x_i the total amount of goods produced in sector i. This would be a practical way to calculate $a_{j,i}$

will denote this quantity by λ_i. We call λ_i *integrated labor cost* (ILC) of the i-th good.[21] Next we will look at how we can calculate the ILCs of all goods. Let's start first with the ILCs of the production goods.

Let's now take a look at an illustrative example of how this would work in a simple economy.

2.3 Simplified example of a neanderthal communal economy

Let's imagine that we belong to a Neanderthal tribe during the Upper Paleolithic era and we want to plan the economy of our goods: stones, sticks and hunting horns. We have the following table of material costs for the realization of each activity:

	Stone carving	Tree felling	Deer hunting
Stones	0.10	0.20	0.20
Sticks	0.00	0.10	0.20
Horns	0.01	0.10	0.50

This table tells us that to carve one stone unit, we need 0.10 stone units, 0.00 sticks units and 0.01 horn units. At first glance this does not seem obvious, for example, stones to make other stones? Horns, what for? Thinking about it more carefully, to carve stones you need other already carved stones to hit them together (hard strikers), and you need horns and sticks as support tools to finish filing the stones (soft strikers).

If we assign indices 1, 2, and 3 to the stone, sticks, and horns respectively, $a_{1,1} = 0.10$, $a_{2,1} = 0.00$, and $a_{3,1} = 0.01$ for the stones; and $a_{1,3} = 0.20$, $a_{2,3} = 0.20$ and $a_{3,3} = 0.50$ for the horns.[22] All these values could be stored in a matrix, so we can perform some calculations with them using tools from linear algebra.

The matrix A could be understood as a "recipe" for producing each type of good:[23]

$$A = \begin{bmatrix} 0.1 & 0.2 & 0.2 \\ 0 & 0.1 & 0.2 \\ 0.01 & 0.1 & 0.5 \end{bmatrix}. \tag{5}$$

[21] Many authors, for example Anwar Shaikh in [25] or Pablo Ruiz Nápoles in [26] define this concept in a different way by including wage compensation to analyze certain phenomena of international trade. In our text this concept would be equivalent to what they call *vertically integrated labor coefficients*.

[22] The $a_{j,i}$ will be referred to later as technical coefficients.

[23] The matrix A will later be called "technological matrix".

Each column of the matrix can be understood as the resources needed by each "industry", in other words, the *demand* in market economies; while each row can be understood as what each sector offers to the others, the *supply*.

One of the Neanderthals of the tribe wants to construct a communal project: she wants to make a series of statues representing motherhood and the Mother Goddess. She claims that she would need 1 stone unit in order to complete the entire project. The question that arises is: how much of each resource should we produce so that we have a final production of 1 stone units?

The extra quantity we want to produce is usually referred to as final demand and can be represented by the final demand vector $\mathbf{d}$, whose components are the final demand d_i for each type of good, in this case, $\mathbf{d} = (1, 0, 0)^T$, because we want to produce only one final unit of the first good (stone). If we wanted to produce two stone units and one unit of horns, the final demand vector would be $\mathbf{d} = (2, 0, 1)^T$.

The quantities of goods we need to produce are an unknown that we need to calculate. This unknown is known as *total production*, and is represented by the vector $\mathbf{x}^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)})^T$ with i being the type of good we want to produce a final[24] unit of ($i = 1$ for the case of stone). This vector stores the quantities $x_j^{(i)}$ of all goods j that are needed to satisfy the final demand of a good i within the system of connected sectors. Beware! These values should not be confused with a_{ji} values, which merely specify the amount of resources needed to produce a good in a particular sector.

Now that we pose the problem separately for each type of good, how much stone would we need to produce? Well, that which is needed to "feed" the production of the interconnected sectors itself *plus* one unit of stone (to satisfy the final demand):

$$\underbrace{x_1^{(1)}}_{\text{Total production}} = \underbrace{a_{11}x_1^{(1)}}_{\text{Sector 1}} + \underbrace{a_{12}x_2^{(1)}}_{\text{Sector 2}} + \underbrace{a_{13}x_3^{(1)}}_{\text{Sector 3}} + \underbrace{d_1}_{\text{Final demand}} . \tag{6}$$

Before we continue, let's analyze the terms present in (6). We see that for example the term $a_{12}x_2^{(1)}$ refers to the amount of stone units needed to produce one sticks unit multiplied by the amount of sticks units needed to produce that 1 of final stone desired. In our particular case we have that

$$x_1^{(1)} = 0.1x_1^{(1)} + 0.2x_2^{(1)} + 0.2x_3^{(1)} + 1 . \tag{7}$$

[24]final refers to the units of goods we will obtain after replacing the goods used in production.

Analogously we obtain for the sticks and horns the following system of linear equations:

$$x_1^{(1)} = a_{11}x_1^{(1)} + a_{12}x_2^{(1)} + a_{13}x_3^{(1)} + 1\,, \tag{8}$$
$$x_2^{(1)} = a_{21}x_1^{(1)} + a_{22}x_2^{(1)} + a_{23}x_3^{(1)} + 0\,, \tag{9}$$
$$x_3^{(1)} = a_{31}x_1^{(1)} + a_{32}x_2^{(1)} + a_{33}x_3^{(1)} + 0\,. \tag{10}$$

In our particular case, we have that

$$\underbrace{\begin{matrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{matrix}}_{\mathbf{x}^{(1)}} \begin{matrix} = \\ = \\ = \end{matrix} \underbrace{\begin{matrix} 0.1x_1^{(1)} + 0.2x_2^{(1)} + 0.2x_3^{(1)} \\ 0x_1^{(1)} + 0.1x_2^{(1)} + 0.2x_3^{(1)} \\ 0.01x_1^{(1)} + 0.1x_2^{(1)} + 0.5x_3^{(1)} \end{matrix}}_{A\mathbf{x}^{(1)}} \begin{matrix} + \\ + \\ + \end{matrix} \underbrace{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}}_{\mathbf{d}}\,. \tag{11}$$

Solving this system of linear equations we would obtain the necessary quantities of each resource to be able to complete the communal project. In matrix form we have that

$$\mathbf{x}^{(1)} = A\mathbf{x}^{(1)} + \mathbf{d}\,. \tag{12}$$

Considering the method described in Section 4.2 for calculating the inverse of a matrix we have that

$$(I - A)^{-1} = \begin{bmatrix} 1.12 & 0.31 & 0.57 \\ 0.01 & 1.16 & 0.47 \\ 0.02 & 0.24 & 2.10 \end{bmatrix},$$ [25]

and so we finally obtain that

$$\mathbf{x}^{(1)} = (I - A)^{-1}\mathbf{d} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 & 0.2 \\ 0 & 0.1 & 0.2 \\ 0.01 & 0.1 & 0.5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$$
$$= \begin{bmatrix} 0.9 & -0.2 & -0.2 \\ 0 & 0.9 & -0.2 \\ -0.01 & -0.1 & 0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 1.12 & 0.31 & 0.57 \\ 0.01 & 1.16 & 0.47 \\ 0.02 & 0.24 & 2.10 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.12 \\ 0.01 \\ 0.02 \end{bmatrix}.$$

where the matrix $(I - A)^{-1}$ is "Leontief's inverse matrix".

Problem solved! We must have a total production of 1.12 stone units, 0.01 sticks units and 0.02 horn units to produce a final unit of stone. This result reflects the need

[25] The more skeptical reader can verify that indeed $(I - A)(I - A)^{-1} = (I - A)^{-1}(I - A) = I$.

to produce a small amount of sticks and horns for this communal work. But didn't it say on the table that no sticks were needed to carve the stone? This is because we need horns to carve stone and to obtain horns we need to use sticks.

Suppose that to carve one unit of stone requires, on average, one hour of labor, to cut one sticks unit requires, on average, two hours of labor, and to hunt one unit of antlers requires, on average, three hours of labor. Then, it would turn out that the integrated labor cost (ILC) of the type of good 1 (stone) would be (approximately) $1.12 \times 1 + 0.01 \times 2 + 0.02 \times 3 = 1.2$ labor hours. So, directly or indirectly, it would take (approximately) 1.2 labor hours to produce one final unit of stone.

2.4 General treatment for industrial economies

As we said, the above would be an example for a simplified economy, but in reality, the reasoning does not change that much when scaling up. The same method can be used to solve the logistical problems of an economy with millions of different items. Illustrating this, however, can only be done using abstract mathematical notation, so the reader will have to process some equations in this section. Also, and unlike Leontief, we have divided the development into production goods and consumption goods for the sake of generality and also to be able to decouple certain operations in different systems of equations. This is inspired by the methods employed by M. Morishima in [27].

2.4.1 Integrated Labor Costs of Producing Goods

We know that in order to produce a good of type 1, $a_{1,1}, \ldots, a_{n,1}$ units of the production goods $i = 1, \ldots, n$ respectively are required. These required units also need inputs in order to be manufactured, and those new inputs will require other inputs, and so on. We must "follow the input trail" until there are no more indirect inputs to be taken into account.

Given the interconectedness between the manufacturing industries, it can be sensed that an increase in one unit of final output of the 1-type good, produces a "multiplicative" or "cascade" effect on the quantity of output goods of all types, even the first type (e.g., to produce electricity, a minimum quantity of electricity is needed). To obtain the total quantities of production goods $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$ that are required to obtain a final unit of good type 1, after taking all the implications into account, we must solve the following system of linear equations:

$$\begin{cases} x_1^{(1)} = a_{1,1}x_1^{(1)} + a_{1,2}x_2^{(1)} + \ldots + a_{1,n}x_n^{(1)} + 1 \\ x_2^{(1)} = a_{2,1}x_1^{(1)} + a_{2,2}x_2^{(1)} + \ldots + a_{2,n}x_n^{(1)} + 0 \\ \vdots \\ x_n^{(1)} = a_{n,1}x_1^{(1)} + a_{n,2}x_2^{(1)} + \ldots + a_{n,n}x_n^{(1)} + 0 \end{cases} \tag{13}$$

Once solved, the ILC of the 1-type good is given by

$$\lambda_1 = \sum_{j=1}^{n} \ell_j x_j^{(1)} .$$

The notation $\sum_{j=1}^{n} f(j)$ represents $f(1)+f(2)+f(3)+\ldots+f(n)$ with f being any function. That is, $\sum_{j=1}^{n} f(j)$ means to sum up all values of the function f in consecutive numbers from the lower bound ($j = 1$) to the upper bound ($j = n$). For example, consider $f(j) = j$, $\sum_{j=1}^{3} f(j) = \sum_{j=1}^{3} j = 1 + 2 + 3 = 6$, i.e., the sum of consecutive numbers from 1 to 3. The symbol $\sum$ is called the summation.

Considering that producing each of the inputs requires a certain period of time, the $\sum_{j=1}^{n} a_{i,j} x_j^{(1)}$ with $i = 1, \ldots, n$ have to be available before the beginning of that period and the $x_i^{(1)}$ with $i = 1, \ldots, n$ will be available at the end of that period. In the actual production of goods, industries use the necessary production goods while at the end of the period they are replaced. The parts of the final good that remain after replacement are called final goods. At the present time we have only one unit of the 1-type good as final good. Similarly we proceed with the good of type 2; to obtain the total quantities of production goods $x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}$ that are required to obtain a final unit of the good type 2. After taking all the implications into account, we must solve the following system of linear equations:

$$\begin{cases} x_1^{(2)} = a_{1,1}x_1^{(2)} + a_{1,2}x_2^{(2)} + \ldots + a_{1,n}x_n^{(2)} + 0 \\ x_2^{(2)} = a_{2,1}x_1^{(2)} + a_{2,2}x_2^{(2)} + \ldots + a_{2,n}x_n^{(2)} + 1 \\ \vdots \\ x_n^{(2)} = a_{n,1}x_1^{(2)} + a_{n,2}x_2^{(2)} + \ldots + a_{n,n}x_n^{(2)} + 0 \end{cases} \tag{14}$$

Once solved, the ILC of the good of type 2 is given by

$$\lambda_2 = \sum_{j=1}^{n} \ell_j x_j^{(2)}.$$

Repeating the process in an analogous way with the other types of production goods, that is, with those of type i being $i = 3, \ldots, n$ we obtain n systems of equations that we can write in matrix form as:

$$X_I = A_I X_I + I \tag{15}$$

with

$$X_I = \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & x_1^{(3)} & \dots & x_1^{(n)} \\ x_2^{(1)} & x_2^{(2)} & x_2^{(3)} & \dots & x_2^{(n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n^{(1)} & x_n^{(2)} & x_n^{(3)} & \dots & x_n^{(n)} \end{bmatrix},$$

$$A_I = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n} \end{bmatrix}.$$

Once the (final) outputs $x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}$ (i-th column of X_I) required to produce one unit (of final output) of the i-th type of output good are determined, we can calculate the ILC of the i-th good thus

$$\lambda_i = \sum_{j=1}^{n} \ell_j x_j^{(i)} \, . \tag{16}$$ [26]

The matrix A_I is the *technological matrix* whose entries are commonly called *technical coefficients* and represent how many units are required on average to produce one unit of each good. Let's look at the columns of A_I, which represent the requirements (inputs) of each type of good.

If we write $L_I = [\ell_1, \dots, \ell_n]$ and $M_I = [\lambda_1, \dots, \lambda_n]$ we have that

$$M_I = L_I X_I \, . \tag{17}$$

So far we have calculated the ILCs of all the production goods. now we will proceed to calculate the ILCs of the consumption goods.

[26] In [28] it is proved that this procedure is equivalent to the more intuitive one of solving the following system of linear equations:

$$\begin{cases} \lambda_1 = a_{1,1}\lambda_1 + a_{2,1}\lambda_2 + \dots + a_{n,1}\lambda_n \\ \lambda_2 = a_{1,2}\lambda_1 + a_{2,2}\lambda_2 + \dots + a_{n,2}\lambda_n \\ \vdots \\ \lambda_n = a_{1,n}\lambda_1 + a_{2,n}\lambda_2 + \dots + a_{n,n}\lambda_n . \end{cases}$$

2.4.2 Integrated Labor Costs of Consumer Goods

The production of consumer goods can be divided into two stages; in the first stage the required production goods are manufactured and in the second stage these goods are combined to obtain the final consumer goods. The amount of production goods required to produce one unit of consumer good of type i (with $i = n+1, \dots, m$) are $a_{1,i}, \dots, a_{1,n}$, for the purpose of replacing these production goods consumed in the manufacturing process, we need $x_1^{(i)}, \dots, x_n^{(i)}$ units of production goods of type $j = 1, \dots, n$ respectively which are determined by the following system of linear equations:

$$\begin{cases} x_1^{(i)} = a_{1,1}x_1^{(i)} + a_{1,2}x_2^{(i)} + \dots + a_{1,n}x_n^{(i)} + a_{1,i} \\ x_2^{(i)} = a_{2,1}x_1^{(i)} + a_{2,2}x_2^{(i)} + \dots + a_{2,n}x_n^{(i)} + a_{2,i} \\ \vdots \\ x_n^{(i)} = a_{n,1}x_1^{(i)} + a_{n,2}x_2^{(i)} + \dots + a_{n,n}x_n^{(i)} + a_{n,i} \end{cases} \tag{18}$$

In the first stage society consumes $\sum_{j=1}^{n} \ell_j x_j^{(i)}$ hours of labor, while in the second stage society consumes ℓ_i hours of labor. Therefore, the ILC of the consumption good of type i is $\sum_{j=1}^{n} \ell_j x_j^{(i)} + \ell_i$.[27] Here, we have performed the procedure for an arbitrary $i = n+1, \dots, m$ type of good. Putting the equations we have obtained so far into matrix form we get:

$$X_{II} = A_I X_{II} + A_{II}\,, \tag{20}$$
$$M_{II} = L_I X_{II} + L_{II}\,, \tag{21}$$

being

$$X_{II} = \begin{bmatrix} x_1^{(n+1)} & x_1^{(n+2)} & x_1^{(n+3)} & \dots & x_1^{(m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n^{(n+1)} & x_n^{(n+2)} & x_n^{(n+3)} & \dots & x_n^{(m)} \end{bmatrix},$$

$$A_{II} = \begin{bmatrix} a_{1,n+1} & a_{1,n+2} & \dots & a_{1,m} \\ a_{2,n+1} & a_{2,n+2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,n+1} & a_{n,n+2} & \dots & a_{n,m} \end{bmatrix}$$

[27] In [28] it is proved that this procedure is equivalent to the more intuitive one of solving the following system of linear equations;

$$\begin{cases} \lambda_{n+1} = a_{1,n+1}\lambda_1 + a_{2,n+1}\lambda_2 + \dots + a_{n,n+1}\lambda_n \\ \lambda_{n+2} = a_{1,n+2}\lambda_1 + a_{2,n+2}\lambda_2 + \dots + a_{n,n+2}\lambda_n \\ \vdots \\ \lambda_m = a_{1,m}\lambda_1 + a_{2,m}\lambda_2 + \dots + a_{n,m}\lambda_n \end{cases} \tag{19}$$

and $M_{II} = [\lambda_{n+1}, \ldots, \lambda_m]$, $L_{II} = [\ell_{n+1}, \ldots, \ell_m]$. We note that X_{II} is a matrix with n rows and $m-n$ columns.

Now, if instead of wishing to produce a final unit of the $i-$th production good, we wish to produce d_i final units of the $i-$th type of good with $i = 1, \ldots, n$ then we need $x_1, \ldots, x_n$ units of each type of production good respectively, which are determined by

$$\underbrace{\mathbf{x}}_{\text{Total output}} = \underbrace{A_I\mathbf{x}}_{\text{Intermediate consumption}} + \underbrace{d}_{\text{Final output}} \iff (I - A_I)\mathbf{x} = \mathbf{d}\,,$$

with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

Therefore, if $(I - A_I)^{-1}$ exists (this matrix is called *Leontief's Inverse Matrix*).[28] [29]

If instead of wishing to produce a final unit of the $i-$th consumption good, we wish to produce d_i final units of the i-th type of good with $i = n+1, \ldots, m$ then we need $x_1, \ldots, x_n$ units of each type of production good respectively, which are determined by

$$\mathbf{x} = A_I x + A_{II}\mathbf{d} \quad \iff \quad , (I - A_I)\mathbf{x} = A_{II}\mathbf{d}$$

being

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} d_{n+1} \\ d_{n+2} \\ \vdots \\ d_m \end{bmatrix}.$$

[28] Investigation on sufficient conditions for the existence of $(I - A_I)^{-1}$ such that the vector $x = (I - A_I)^{-1}d$ has all nonnegative entries is beyond the scope of the article, the reader interested in questions related to the above problems may consult [28, Chapter 2] and [27].

[29] Under certain technical conditions (the reader may consult, for example, [29, p.351]) on A_I, which are generally satisfied in an economy, it follows that $(I - A_I)^{-1} = I + A_I + \ldots + A_I^q + A_I^{q+1} + \ldots$ (a limit would appear here but defining it rigorously escapes the purpose of the article, instead, the reader may interpret it as an 'infinite sum'). In fact, if $A_I^{q+1} = 0$ then $(I - A_I)^{-1} = I + A_I + \ldots + A_I^q$. These equalities are really interesting, beyond giving us alternative ways to calculate $(I - A_I)^{-1}$ they allow us to give a precise economic sense to the Leontief Inverse Matrix. It suffices to understand that $A_I\mathbf{x}$ are the inputs required to produce x_i units of the type of good i with $i = 1, \ldots, n$, $A_I^2\mathbf{x}$ are the inputs required to produce the inputs required to produce x_i units of the type of good i with $i = 1, \ldots, n$ and so on then $\mathbf{x} = (I - A_I)^{-1}\mathbf{d}$.

Therefore, if $(I - A_I)^{-1}$ exists, then $\mathbf{x} = (I - A_I)^{-1} A_{II} \mathbf{d}$.

Now we can clearly see the basic strategy of our economic planning. Having fixed the vector d, which represents the final quantity of goods of each type that we wish to produce, we calculate how many units in total of each type of production good, x, we need to manufacture. Thus, we can avoid the two major problems of the Soviet Union described previously in the Introduction, without the need to use a monetary unit to describe labor flows.[30][31] Of course, estimating d poses a challenge as well, although much less of a challenge than it might seem at first glance, and even more so with today's technology, which allows us to have real-time mechanisms for information feedback. This problem is discussed extensively in [19].

[30] Although obtaining the inverse of large matrices is a relatively costly process (since the number of operations does not grow linearly with the number of rows, as we prove in Section 4) the reader should have into account that:

1. The matrices A_I and A_{II} are not just any matrices. On the one hand, their size depends on the level at which the economy is disaggregated into types of items (or sectors). On the other hand, the more it is disaggregated, the sparser the matrices become (i.e., there are more entries in the matrix that are null). Sparse matrices are very special because they are highly manageable. Furthermore, the total type of goods in a real economy will always be considerable less than the total population.
2. We do not need to calculate anything exactly. We only need iterative methods that provide approximate solutions with the desired precision. The are of mathematics that is responsible for this is called Numerical Analysis and it has been one of the most fruitful areas of applied mathematics in the last century, even more so with the development of computing. The reader interested in this type of tools can consult [30-34].

[31] If personal consumption items in public stores were to be purchased against their equivalent ILC in labor tokens, the problem described in Appendix A would disappear.

3 Optimization

This would answer many of the doubts about the functioning of cyber-socialist planning, but in a complex economy there are more problems to take into account.

Given a particular agro-industrial complex, different ways of approaching the same problem, task or production objective are always conceivable, each with partially or entirely different results. We are referring here to alternative options when it comes to choosing technologies, workforce distribution, transport and/or delivery routes, etc. Let us think of an easy example: toiling the land. A priori, one would think that plowing the field with a tractor is more efficient than using hand tools. But, if we think about it more carefully, this dichotomy is not realistic. Tractors don't just fall out of the sky. They also have to be produced, and one could think of scenarios in which it would be unreasonable to start assembling tractors. This example is as simple as it gets, but with a little imagination, it is not difficult to realize that many of the challenges of this century will involve situations of this kind: how do we make the transition to a post-carbon economy? Do we invest millions and millions of man-hours in an experimental energy that, after a few years, could solve the energy problem for centuries, or do we play it safe and simply combine, as best as we can, the existing ones?[32]

In any case, these problems can be summarized as the attempt to maximize or minimize certain quantities under certain constraints. Among other things, one could try to minimize the number of working hours required in a dangerous and/or unpleasant sector or the amount of CO_2 emitted to produce a certain product. Well, thanks to the contributions of the economist and mathematician Leonid Kantoróvich,[33] we know that most of these problems can be modeled mathematically in what are known as *optimization problems*. Formalizing and solving these, taking into account the available resources, gives us clear indications of how to update our economy efficiently.

Let us recall what was said in section 1. In the market ecosystem, social needs are eclipsed by the mediation of the imperative of profitability; these needs are interpreted as opportunities for profit, subordinating all technical innovations of our collective knowledge to this end. This dynamic is tremendously arbitrary and gives rise to endless problems (overproduction of unnecessary but profitable products, shortage of other necessary but unprofitable ones, crises, etc.), but.... isn't there another way of doing things? Isn't it intuitive that the optimization mechanisms used today by Amazon or

[32] This is not to be confused with the problem, mentioned in the previous chapter, of calculating the integrated labour cost of a good that is being produced *simultaneously* in different ways. The question now is to select, among a variety of *possible* initiatives, *the best one*.

[33] Kantoróvich was also awarded the Bank of Sweden's Alfred Nobel Memorial Prize in Economic Sciences for this topic in 1975.

Walmart,[34] on an intra-company scale, could be used for emancipatory purposes, even more efficiently, by socializing all companies and abolishing the enormous constraint of hiding information from each other?

In this chapter we will see just that: how mathematical optimization could be used in a democratically planned economy to ensure the highest possible performance of our infrastructures. As explained elsewhere [36] [37], this does not imply that all our public enterprises have to use the same productive techniques. There is enormous room for innovation and experimentation, but this is a topic beyond the scope of this article. Let us now focus on the idea of optimization.

3.1 Mathematical Optimization

The first step in modeling an optimization problem is to define the *objective function* to maximize or minimize. Put mathematically, we need to find a function $f : \mathcal{A} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers (any number) and $\mathcal{A}$ is normally a subset of the n-dimensional Euclidean space $\mathbb{R}^n$. The latter may sound a bit complex, but it can be easily understood with an example. We are all used to 3 dimensions in our everyday life (width, height and length). Or even 4 dimensions! If we have heard of Albert Einstein... Well, these 3 dimensions represent a Euclidean space $\mathbb{R}^3$. On the other hand, $\mathcal{A}$ is a subset of this space, i.e., all elements of $\mathcal{A}$ belong to $\mathbb{R}^3$, so graphically it would be a bounded portion of it. For example, a sphere would be a subset of the Euclidean space $\mathbb{R}^3$.

Finally, all that remains is to extend the mathematics used for $\mathbb{R}^3$ to n dimensions. The points in $\mathbb{R}^3$ are defined by 3-coordinates, i.e., they are nothing but ordered triplets (3-tuples) of real numbers. Analogously, the points in $\mathbb{R}^n$ are nothing but $n-$tuples of real numbers, i.e., each point in $\mathbb{R}^n$ is defined by n ordered numbers. Hopefully, this will sound familiar to the reader, as the elements of $\mathbb{R}^n$ are exactly the row or column vectors with n columns or n rows respectively introduced in Section 2.

This last step is simple but hard to imagine, since no one has ever been able to observe an object in more than 3 dimensions. However, in mathematics it is perfectly possible. In economics, the n dimensions have a more tangible meaning, since they refer to the n types of products whose production is to be optimized, for example, by representing the units of each type of item to be used in a certain task.

The subset $\mathcal{A}$ will be defined by the *restrictions* of the problem. Formally, the points of $\mathcal{A}$ will be $n-$tuples of numbers that will verify certain inequalities and/or equalities. For example, these constraints may come from limitations on the total number of

[34] You can read about the use of these techniques in these corporations in Chapter 4 [35].

working days employed or the total number of tons of CO_2 generated.

At this point, we already know all the ingredients needed to describe an optimization problem, i.e., the objective function and the constraints. Optimization is a very broad branch of mathematics, which involves a myriad of different techniques depending on the specific characteristics of the problem to be solved. In this article we focus on *linear programming*. As its name indicates, it refers to those optimization problems composed only of linear functions. Section 3.2 presents in detail the description and application of linear programming to economic problems. However, certain aspects of the economy, such as economies of scale, cannot be dealt with by linear programming, because either the objective function is not linear or the relationships defining the constraints are not linear. This aspect is discussed in more detail in Section 3.3.

3.2 Linear Programming

Linear relationships between variables are relationships that conserve proportions. Consequently, a linear function f defined on $\mathbb{R}^n$ sends linear combinations of the vectors x, y into linear combinations of $f(x)$, $f(y)$, with the same proportionality constants, formally, we will say that $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all $\alpha, \beta \in \mathbb{R}$ and for each x, y in $\mathbb{R}^n$. The appearance that these functions have is, for example, that of a straight line in the plane or that of a plane in $\mathbb{R}^3$. These types of functions are very tractable by computers and have interesting properties.

Linear programming problems are usually represented in the literature with the following notation:

$$\begin{aligned} \max_{x} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{22}$$

where $\mathbf{c}^T$ is a row vector $1 \times n$, $\mathbf{x}$ is a column vector $n \times 1$, $\mathbf{A}$ a matrix $m \times n$ and $\mathbf{0}$ is a column vector $n \times 1$ whose elements are only 0's.

The previous problem tries to maximize the (linear) function $\mathbf{x} \to \mathbf{c}^T\mathbf{x}$ subject to certain restrictions that we will explain below.[35] The two apparent constraints, being expressed in matrix notation, actually correspond to multiple individual constraints.

[35] As we stated in 6 every linear application can be represented by a function of the form $\mathbf{x} \to A\mathbf{x}$ being A a certain matrix. Therefore, in the above problem the objective function can be any linear function

Therefore, we find it convenient to use the matrix notation in this type of problems.

As we saw in section 2, when the concept of matrix-vector product was introduced, $A\mathbf{x} \leq \mathbf{b}$ would represent m inequalities, since the matrix A has dimensions $m \times n$ where n is the number of variables in our problem. The m constraints can refer, for example, to different inputs in a manufacturing process, hours worked, machinery used.... What we need to keep in mind is that constraints will always have a meaning in the real world. The second constraint is quite common and ensures that the variables in $\mathbf{x}$ do not take negative values to avoid solutions that do not make sense in the real world, such as producing a negative number of vehicles.

Let us now try to solve a concrete problem with the tools we have so far. It seems that in the latest plebiscite on the de-carbonization of the economy, citizens have chosen to give more importance to the bicycle as a means of transportation. Unlike in a capitalist economy, it is not necessary to wait for supply and demand adjustments or for some capitalist to detect a market opportunity, but the result of the plebiscite can be implemented directly in the entire bicycle sector.

Given a bicycle factory that produces mountain bikes (x_1) and electric bikes (x_2), the objective is for the factory to maximize its production taking into account that the purchases of electric bikes are expected to be twice as high as those of mountain bikes, since the former are much more comfortable for commuting long distances to work. These types of preferences can be reflected in the objective function using the vector $\mathbf{c}^T = [1 \quad 2]$, resulting in the objective function $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} = x_1 + 2x_2$. In other words, electric bicycles have a higher weight in the function and will therefore be prioritized in production.

The weekly inputs to the factory are 60kg of steel and 180kg of aluminum. The production of each mountain bike requires 1kg of steel and 4kg of aluminum, while the electric bikes require 2kg of steel and 2kg of aluminum. This results in the constraints for steel (equation (23)) and aluminum (equation (24)), since production can never exceed the raw materials needed in their manufacture. Finally, the working hours required for each bicycle must be considered. The factory consists of 4 employees working 35h per week each and each of them takes 3 hours to finish a mountain bike and 4 hours for an electric one (equation (25)).

$$1x_1 + 2x_2 \leq 60 \tag{23}$$

$$4x_1 + 2x_2 \leq 180 \tag{24}$$

$$3x_1 + 4x_2 \leq 140 \tag{25}$$

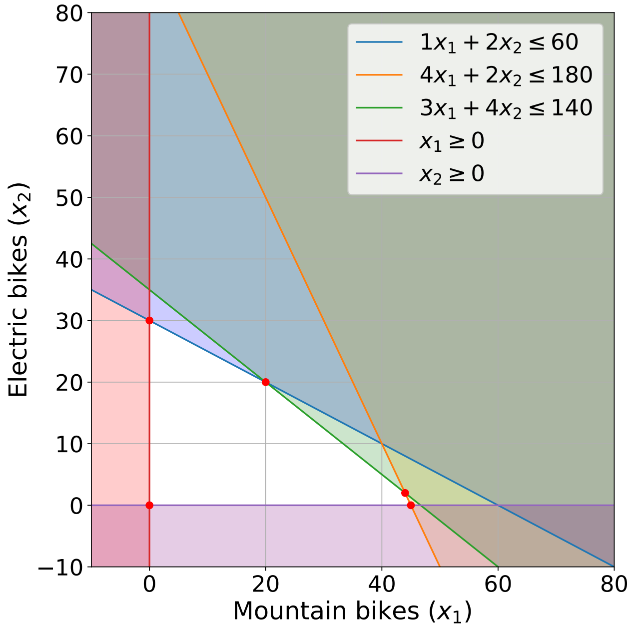

Figura 3: The feasible region of the optimization problem is a polyhedron. The different colored planes correspond to the constraints of steel (blue), aluminum (orange), working hours (green) and non-negativity of the solution (red and violet), respectively.

Equation 26 shows the constraints in matrix form. Note that the last two rows of the matrix A and the vector $\mathbf{b}$ ensure that the number of bicycles is not negative.

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 4 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \mathbf{b} = \begin{bmatrix} 60 \\ 180 \\ 140 \\ 0 \\ 0 \end{bmatrix} \tag{26}$$

The feasible region is shown in Figure 3, where the shaded areas represent the space in which the constraints are not satisfied. Between the intersections of all the constraints, there remains a polyhedron (white region in the figure) such that any point within this polyhedron satisfies all the constraints. It can be shown that the point that maximizes (or minimizes) the objective function must be one of the vertices of this polyhedron, so that any algorithm that seeks to find the maximum (or minimum) of the objective function, must evaluate it in all or some of these vertices to solve the problem. One of the best known algorithms that solve this problem is the "simplex" algorithm.

3.2.1 The simplex method and its applications

In the following, we will state, intuitively and without going into the details of the calculation, the modus operandi of the *simplex algorithm*, the method commonly used to solve linear programming problems.

To understand it, it is enough to know two characteristics of the problem: one related to the constraints and the other to the objective functions.

The first is the specific shape of the feasible region in which the solutions meet the requirements imposed by the constraints. This is a convex polyhedron (a cube or a dodecahedron are two examples, although feasible regions need not be only 3-dimensional nor be regular).[36]

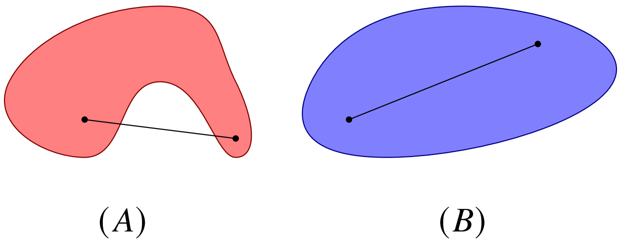

Figura 4: Visual example of a non-convex set (A) and a convex set (B).

The second feature of the problem is that the objective function, being also a linear function, orders the space of solutions by dividing them into lines, planes or hyperplanes (i.e., the extension of the concept for more than 3 dimensions) in which the function has the same value (see figure 6).

With this in mind, the idea is to make the objective function increase (or decrease). To do this we would move up (or down) in the direction perpendicular to these planes. These two results together lead us to the following conclusion: the optimal solutions can only be at one vertex or at several ones (along with the points between them: an edge or a face of the polyhedron). This can be pictured in 3D by resting a polyhedron on a table either on one point or more, and looking for the highest point: intuitively we see that it will indeed be a vertex (or more).

[36]Before proceeding, please remember that, mathematically, a set is convex if and only if, for any two points of the set we take, the segment joining them is contained in said set.

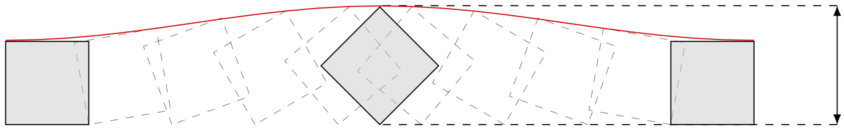

Figura 5: Visual example of the possible maxima of a simple convex polyhedron (square) where $h(\alpha)$ is the height as a function of the angle of inclination. The highest points are either a vertex or an edge.

As we are interested in finding an optimal feasible solution, we do not care if there are several more, the crucial part is to arrive at one of them. What makes the previous result important is that our optimum is going to be in some vertex. With this condition in mind, an algorithm was designed that uses this result: the simplex method. That proceeds in three phases: 1. initialization, 2. loop and 3. finalization.

1. The algorithm is initialized by taking any vertex. From there, we analyze its adjacent vertices (connected to it by an edge) and calculate in which of them the objective function has a larger value (if we are maximizing).

2. Then, we move to that new vertex and repeat the process. By always increasing the value of the objective function we will eventually reach the optimal vertex.

3. This is confirmed by verifying that its adjacent vertices have a lower value for the objective function. There, the simplex ends having reached the optimal feasible solution.

Once we know the simplex method, we can apply it to the above problem for the optimization of bicycle manufacturing to obtain the solution of the problem. The vertices of the polyhedron that conforms the feasible region are marked as red dots in Figure 3. These vertices are the points at which the planes forming the constraints intersect. For example, to obtain the cut point of the constraints $x_1 \geq 0$ and $1x_1 + 2x_2 \leq 60$ it is only necessary to solve the system of equations formed by the two constraints. In other words, we substitute $x_1 = 0$ of the first constraint in the equation $1x_1 + 2x_2 = 60$ to obtain the cut-off point $(x_1, x_2) = (0, 30)$. Proceeding in a similar way with the rest of the constraints, the list of vertices $[(0, 0), (0, 30), (20, 20), (44, 2), (45, 0)]$ is obtained.

The reader need only substitute each of these points into the objective function $x_1 + 2x_2$ to find its maximum, which in this case is 60. It is interesting to note that this maximum could be reached in two different vertices (i.e., for $(x_1, x_2) = (0, 30)$ and $(x_1, x_2) = (20, 20)$). In principle, either would be a valid solution to our problem and the final decision could be made, for example, randomly (e.g., by flipping a coin) or based on coordination at the sectoral scale (it could happen that the demand for mountain bikes wouldn't be met if $x_1 = 0$ were chosen).

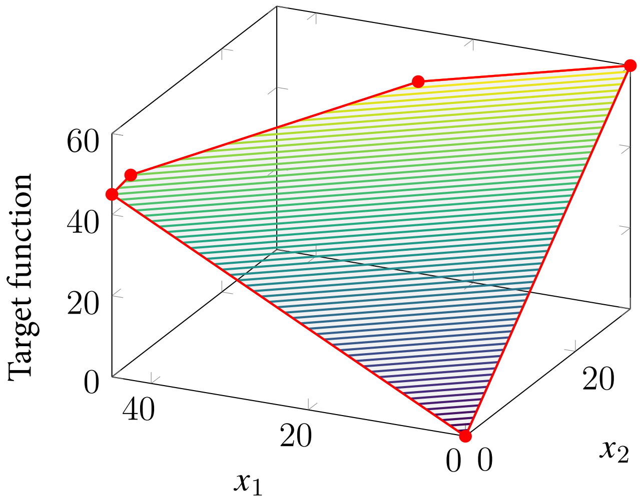

Figura 6: Values of the objective function in the feasible region, represented by the area bounded by the red lines. Note that the function has a constant value on the shaded lines inside the polygon (lower value violet, higher value yellow).

For the reader interested in the formalism upon which this algorithm rests, we recommend consulting the corresponding bibliography [38].

Let us see, with a series of examples, how versatile linear optimization is. To do so, we will see that a great variety of problems - in our case economic ones - are in fact linear programming problems in one form or another, and are therefore solvable by methods such as the simplex described previously.

3.2.2 Historical example from the Plywood Trust Central Laboratory

We will discuss an example that is, in fact, based on a real problem that the Plywood Trust Central Laboratory presented to L. Kantorovich in 1939 (described in [39]).

The problem was to maximize the production of different types of wood in certain proportions by making use of certain machines. They had 8 machines to produce lumber and 5 different types of lumber. The request was to ensure that 10%, 12%, 28%, 36% and 14% of the final product would be of the first, second, third, fourth and fifth types of wood respectively. Let p_k be the proportion of wood of type k required to be produced, in our case: $p_1 = 0,1$, $p_2 = 0,12$, $p_3 = 0,28$, $p_4 = 0,36$ y $p_5 = 0,14$. Let us denote by $\alpha_{i,k}$ the number of units of wood of type k that are produced in a working day using machine i and by $h_{i,k}$ the time, expressed as a fraction of the working day, that we are going to use machine i to produce wood of type k. The $\alpha_{i,k}$ are data known to the Central Laboratory, that represent the productivity of the machines at producing the different types of wood. These data were specifically as follows:

Machine number	Kind of lumber				
	1	2	3	4	5
1	4,0	7,0	8,5	13,0	16,5
2	4,5	7,8	9,7	13,7	17,5
3	5,0	8,0	10,0	14,8	18,0
4	4,0	7,0	9,0	13,5	17,0
5	3,5	6,5	8,5	12,7	16,0
6	3,0	6,0	8,0	13,5	15,0
7	4,0	6,0	9,0	14,0	17,0
8	5,0	7,0	10,0	14,8	18,0

Cuadro 2: The $\alpha_{i,k}$ with $i = 1, 2, 3, 4, 5, 6, 7, 8$ and $k = 1, 2, 3, 4, 5$.

The problem consists in obtaining the $h_{i,k}$ with $i = 1, \ldots, 8$ and $k = 1, \ldots, 5$ that verify the following conditions:

1. $h_{i,k} \geq 0$.[37]

2. $\sum_{k=1} h_{i,k} = 1$.[38]

3. $\frac{1}{p_1} \sum_{i=1}^{8} h_{i,1}\alpha_{i,1} = \frac{1}{p_2} \sum_{i=1}^{8} h_{i,2}\alpha_{i,2} = \frac{1}{p_3} \sum_{i=1}^{8} h_{i,3}\alpha_{i,3} = \frac{1}{p_4} \sum_{i=1}^{8} h_{i,4}\alpha_{i,4} = \frac{1}{p_5} \sum_{i=1}^{8} h_{i,5}\alpha_{i,5}$ and that this common value is maximum.[39]

The solution to the above problem, applying the simplex method, is as follows:

[37] This is obvious but it is important not to obtain solutions with $h_{i,k} < 0$, which are meaningless

[38] Each machine is going to spend the entire working day producing some type of wood

[39] If we denote by $z_k = \sum_{i=1}^{8} h_{i,k}\alpha_{i,k}$ the units produced of each type of wood $k = 1, 2, 3, 4, 5$, then the total units of wood produced (in a working day) are $M = \sum_{k=1}^{5} z_k$. We have that the conditions $z_k = p_k M$ for all $k = 1, 2, 3, 4, 5$ are equivalent with the first conditions of 3. To see this, it suffices to note that $p_1 + p_2 + p_3 + p_4 + p_5 = 1$, by definition of p_k, and that

$$
\begin{aligned}
p_k M &= p_k\left(\frac{p_1 z_1}{p_1} + \frac{p_2 z_2}{p_2} + \frac{p_3 z_3}{p_3} + \frac{p_4 z_4}{p_4} + \frac{p_5 z_5}{p_5}\right) \\
&= p_k(p_1 + p_2 + p_3 + p_4 + p_5)\frac{z_k}{p_k} \\
&= z_k.
\end{aligned}
$$

These latter conditions are clearer than those in 3.

Machine Number	Kind of lumber				
	1	2	3	4	5
1	0	0,3321	0	0	0,6679
2	0	0,9129	0,0871	0	0
3	0.5744	0	0,4256	0	0
4	0	0	0,9380	0,0620	0
5	0	0	1	0	0
6	0	0	0	1	0
7	0	0	0	1	0
8	1	0	0	0	0

Cuadro 3: The optimal $h_{i,k}$, where $i = 1, 2, \ldots, 8$ and $k = 1, 2, \ldots, 5$ the we were looking for.

The optimum total number of units of each type of wood that can be produced in a workday is

$$\sum_{i=1}^{8} h_{i,1}\alpha_{i,1} = 0,5744 \cdot 5 + 1 \cdot 5 = 7.872\,,$$
$$\sum_{i=1}^{8} h_{i,2}\alpha_{i,2} = 0,3321 \cdot 7 + 0,9129 \cdot 7,8 = 9.44532\,,$$
$$\sum_{i=1}^{8} h_{i,3}\alpha_{i,3} = 0,0871 \cdot 9,7 + 0,4256 \cdot 10 + 0,9380 \cdot 9,0 + 1 \cdot 8,5 = 22,04287\,,$$
$$\sum_{i=1}^{8} h_{i,4}\alpha_{i,4} = 0,0620 \cdot 13,5 + 1 \cdot 13,5 + 1 \cdot 14,0 = 28,337 \text{ and}$$
$$\sum_{i=1}^{8} h_{i,5}\alpha_{i,5} = 0,6679 \cdot 16,5 = 11.02035\,,$$

respectively.

3.2.3 Miscellaneous other possible examples

Now suppose we have n machines that are part of the production process of a certain type of good, composed of m parts (in principle there could be repetitions of type of parts, however, for the sake of simplicity, we will treat each part as a unique type so this will not pose any problem). Let us denote by $\alpha_{i,k}$ the number of parts of type k

that are produced in a working day using machine i. Let us note that in case machine i is not able to produce parts of type k (e.g., a tractor could not produce screws) then we would set $\alpha_{i,k} = 0$. So, what are we looking for? We wish to distribute the work among the different machines such that the total number of completed items is maximized. Let us denote by $h_{i,k}$ the time, expressed as a fraction of the working day, where machine i is used to produce parts of type k. Our problem is precisely to determine the $h_{i,k}$ with $i = 1, ...n$, $k = 1, ..., m$ such that we maximize the final number of finished items. Let us see what conditions the $h_{i,k}$ have to verify. It is evident that $h_{i,k} \geq 0$ for all i, k and that for each i:

$$\sum_{k=1}^{m} h_{i,k} = 1 \ .$$ [40]

If z_k is the total number of parts of type k produced we have that

$$z_k = \sum_{i=1}^{n} \alpha_{i,k} h_{i,k} \ ,$$

since $\alpha_{i,k} h_{i,k}$ gives us the total number of parts of type k produced using machine i. If we wish to obtain complete goods the condition $z_1 = z_2 = ... = z_m$ must be imposed, i.e., the total number of items (necessary to the production of this good) of each type must be equal. We must maximize the common value, z, of all these quantities. Therefore solving the stated problem leads us to solve Problem A:

Determine $h_{i,k}$ with $i = 1, ..., n$, $k = 1, ..., m$ such that

1. $h_{i,k} \geq 0$ for all $i = 1, ..., n$, $k = 1, ..., m$.
2. $\sum_{k=1}^{m} h_{i,k} = 1$ for every $i = 1, ..., n$.
3. $z = z_1 = ... = z_m$ and z is maximum where $z_k = \sum_{i=1}^{n} \alpha_{i,k} h_{i,k}$ for all k.

The reader can easily write Problem A in the form (22).[41] Other variations of this problem also fit the structure of Problem A. For example, if we only produced a single type of part with different machines through different processes and there were different ways to produce it,[42] we would arrive at Problem A. With the difference that $\alpha_{i,k}$, in this case, would be the number of parts that have gone through the $k-$th process using machine i during a working day.

We can also add additional limiting conditions to the original problem, for example, if each manufacturing process required a different amount of energy, we might

[40] We can assume that each machine is going to be used the entire working day, otherwise, the condition would be replaced by $\sum_{k=1}^{m} h_{i,k} \leq 1$.

[41] The problem described in 3.2.2 belongs to this family of problems if we take as "$\alpha_{i,k}$" $\frac{\alpha_{i,k}}{p_k}$ being p_k the proportion of wood of type k that is required to be produced.

[42] For example, making cabinets requires first that we fell trees, then cutting the wood into the right dimensions, ... etc., and all these processes could be done with different machinery.

want to limit the total energy expenditure. Let $c_{i,k}$ denote the kWh per day of energy that manufacturing part type k using machine i uses up. The total energy expenditure is given by the expression $\sum_{i=1}^{n}\sum_{k=1}^{m} h_{i,k}c_{i,k}$. We can then add to Problem A the constraint that the total energy expenditure be less than or equal to C, for some fixed C. Thus, we arrive at Problem B: Determine $h_{i,k}$ with $i = 1, ..., n$, $k = 1, ..., m$ such that

1. $h_{i,k} \geq 0$ for all $i = 1, ..., n$, $k = 1, ..., m$.
2. $\sum_{k=1}^{m} h_{i,k} = 1$ for every $i = 1, ..., n$.
3. $z = z_1 = ... = z_m$ and z is maximum being $z_k = \sum_{i=1}^{n} \alpha_{i,k}h_{i,k}$ for all k.
4. $\sum_{i=1}^{n}\sum_{k=1}^{m} h_{i,k}c_{i,k}C$.

We note that $c_{i,k}$ can be replaced by the expenditure of water or labour used in producing the k type of parts using the i machine and thus we can impose constraints on the total amount of water that can be expended or on the total number of people to be employed.

Now, suppose that the same machine is able to produce, at the same time, different parts (or to perform several processes at the same time) and that we can organize the production process using different production methods. Let $\lambda_{i,k,l}$ be the number of parts of type k that are produced under the l-th production method using machine i. If $h_{i,l}$ is the time, expressed as a fraction of the working day, spent using machine i for the l-th production method then the total number of parts of type k produced using all machines, z_k, will be expressed by $z_k = sum_{i,l}\lambda_{i,k,l}h_{i,l}$. The same reasoning as before leads us to Problem C: Determine the $h_{i,l}$ such that

1. $h_{i,l}0$ for all i, l.
2. $\sum_{k=1}^{m} h_{i,l} = 1$ for every i.
3. $z = z_1 = ... = z_m$ and z is maximal being $z_k = \sum_{i,l} \lambda_{i,k,l}h_{i,l}$ for all k.

It is even possible to extend the problem even further and express many other problems in the same way, however, we believe that with these examples the reader will have enough to understand the idea behind its application and how versatile linear programming is when applied to economic problems of any scale. The reader interested in expanding on these topics can consult [39] and [40]. With respect to scale, the solution to the problems we have posed, which basically deal with optimizing the use of machinery, make a greater difference when applied to larger sectors of the economy.

3.3 Non-linearities

So far we have considered that variations in the economy occur in a linear fashion. This means that we assume that to produce n times more (or less) quantity of a given

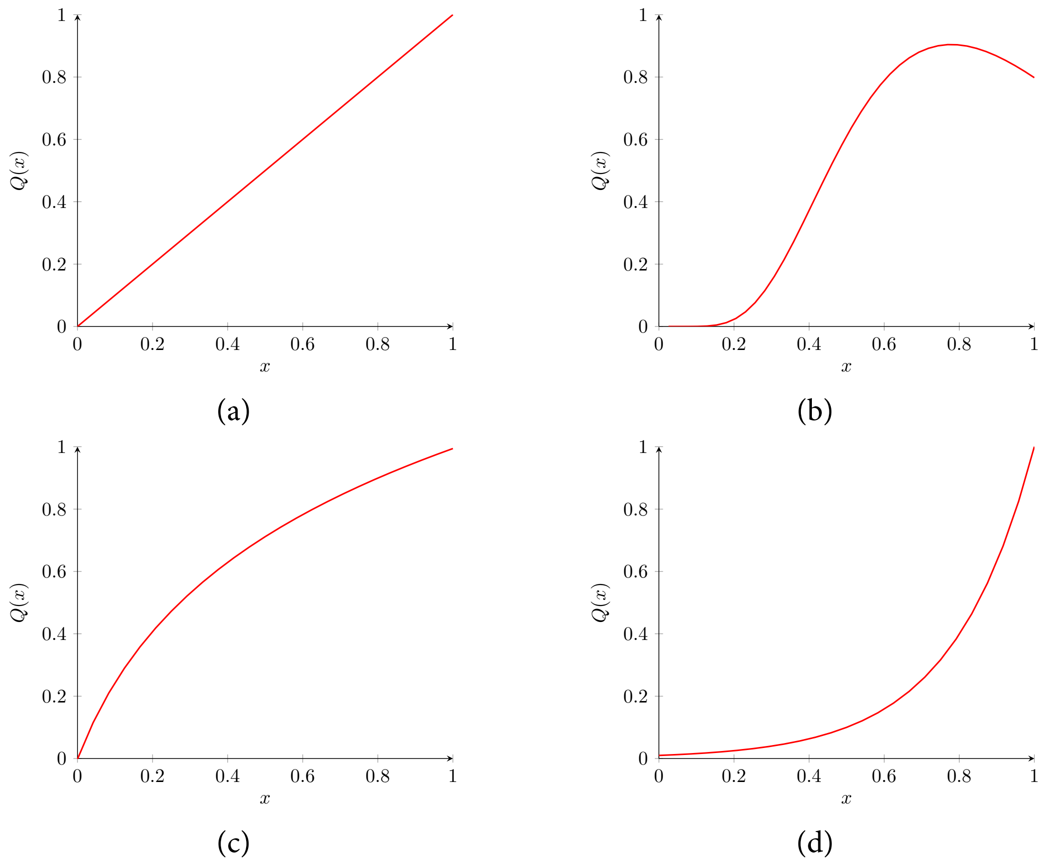

Figura 7: Examples of linear (a), and nonlinear (b,c,d) functions. If we understand them as production functions, the horizontal axis is the amount of inputs and the vertical axis is the amount of outputs.

product x, we will need n times more (or less) inputs. This is the best initial approximation to the actual macroscopic functioning of economies. However, at a more detailed level of analysis we will find that certain sectors behave in a markedly different way: we call these behaviours non-linearities (see figure 7).

There are different types of phenomena that prevent linear behavior: fixed costs, increasing marginal productivity, decreasing marginal productivity, etc... In addition, we would include among the non-linearities those economic situations in which integer results are required.[43] We include them because we are handling a notion of linearity in the strictest sense: in linear cases we can reduce or increase the set of inputs/outputs by an arbitrary ratio, without having to restrict ourselves to integer outcomes.

[43]We're talking about integer variables, which we'll comment on in the following section, where we will clarify both these notions and the distinction with respect to the "arbitrary ratio".

3.3.1 Challenges

To facilitate the intuitive understanding of these challenges, we will present several concrete examples of these phenomena in real life economies:

- Fixed costs: to visualize them, let us imagine we used a linear model for the case of gas extraction, such that we would find a ratio of 2 hours to extract 1L of gas (on average over the whole sector). Considering this, we could conclude that to extract a total of one liter of gas we would only need two hours of human labor. However, as soon as we add a bit of realism to the problem, we realize that this is an absurdity. To extract the first litre of gas we would first have to build huge extraction and transportation facilities that require thousands of labor hours. These investments that we need as a basis to carry out certain production processes and that do not vary with the quantity produced are what we call "fixed costs" and are essential in various sectors.

- Marginal productivities: as examples of increasing and decreasing marginal productivity we have economies and diseconomies of scale. Economies of scale occur when, as the quantity produced increases, the average costs per unit produced are reduced (an example is the capitalist conglomerates that use this advantage in the market). With diseconomies, on the other hand, the opposite is true. In sectors such as mining or agriculture, the most fertile areas with the highest output per hour worked are first cultivated, but, as these are used up, less productive lands are toiled, giving rise to a decreasing marginal productivity. In these cases, if we wanted to double production, we would have to invest more than twice as much work.

- Integer variables: an integer variable $(\ldots, -2, -1, 0, 1, 2, \ldots)$ is distinguished from real variables $(0.1, -34.3, 1, 1, \ldots)$ which we use in linear programming, in that they have no decimal part. Economic examples of integer variables are factories or many of the consumer goods that are produced. For planning purposes, we must take them into account so that the plan does not foresee building one third of a factory at a given site or assigning half a car for such a store, for example.

Next, we will look at the different strategies used to address these difficulties in calculating a feasible and approximately optimal plan.

3.3.2 Solutions

Before going into the types of solutions, let us note something about their use. When having to solve problems beyond the mere linear planning, the computational complexity logically increases, restricting more or less our margin of maneuver [41]. This, however, does not prevent us from applying these methods at scales lower than the directly national one: the exact solutions to the problems posed can be found, in practice, at other scales such as the sector scale, regional or local.[44] There is a rich literature of more complex but very useful methods that could be applied in such cases, lowering the computational complexity by approximations and ensuring practically feasible resolutions. Although an in-depth analysis of these methods is beyond the scope of this introductory article, in the following we briefly describe some examples so that the reader becomes familiar with them.

Piecewise Linearization

To deal with nonlinear functions (such as increasing or decreasing productivities, for example) we can "linearize" them. To linearize in mathematics is to choose a linear function that closely resembles the original, either locally (as in the case of piecewise linearization) or globally.

An intuitive example would be the model we use for the Earth depending on our objective. We all know that the Earth is spherical (technically a spheroid) and if we were to calculate satellite orbits we would be forced to take its sphericity into account. However, when designing a building or a train network, we assume that the Earth is flat without taking into account its sphericity, because it makes the calculations much easier and at this scale its global shape is irrelevant.

To find an approximately optimal plan we can use exactly the same tool, considering that in each local region of the nonlinear functions there is a linear function that approximates them well (piecewise linearization). This tool is used in many industries; for example, we could linearize production costs when we have decreasing marginal costs that have been estimated with some error (we have scattered data, see figure 8).

Non-convex optimization

Non-convex optimization is used precisely when we have increasing marginal productivity behaviors (among others) such as those of economy of scale. This gives rise to non-convex feasible regions where we cannot apply the linear programming methods we have been using.[45]

[44] In addition to the algorithmic approach presented in this paper, it is worth highlighting the treatment of complexity in cybernetics. In the postwar USSR, the modeling of dynamic systems through continuous feedback gained much relevance precisely in the context of regional and local planning. As a visible face of this initiative, it is worth mentioning the little-known group of Soviet cyberneticians at the Novosibirsk Institute of Economics, Siberia [42].

[45] The concrete mathematical mechanism by which these types of behaviors, in certain sectors, give rise to non-convex feasible regions involves the second derivatives of their production functions. However, both this mechanism and its translation into actual economic behaviors are beyond the explanatory scope of this paper.

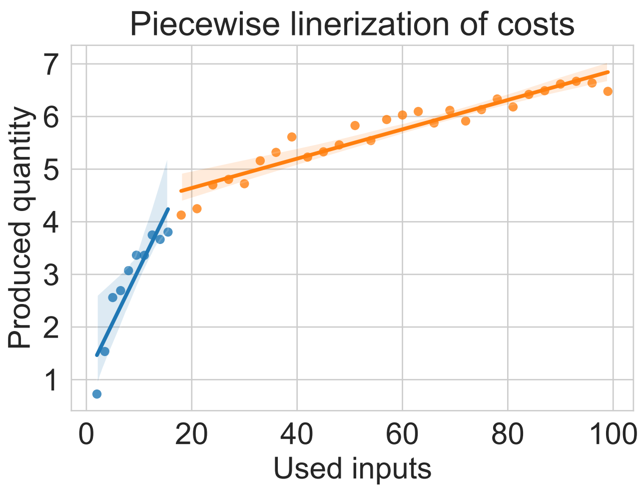

Figura 8: Example of decreasing marginal costs. We approximate the data by two straight lines, depending on whether more or less quantity is produced. The shaded area indicates the confidence interval of the approximation.

However, one tool that can be used to solve this problem is to inscribe a convex polyhedron P_1 inside the non-convex region P so that we can apply the convex programming tools there. By obtaining an approximate maximum x_1 inside that region we can use it to construct a new convex region P_2 that is closer to an optimum of the non-convex region x_l (at least a local one) and repeat the same process again (see figure 9). That way, by recurrence we would end up approaching (with a small margin of error) an optimum [21] (at least a local one).

In fact, this is how today's capitalist market unconsciously works: approximating optima by recurrence. The good news is that in a planned economy, having a global

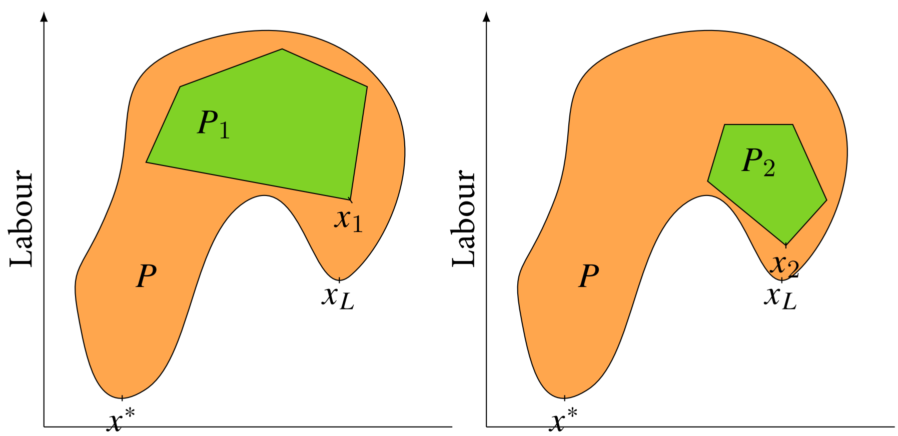

Figura 9: Graphs representing the first two iterations of the method

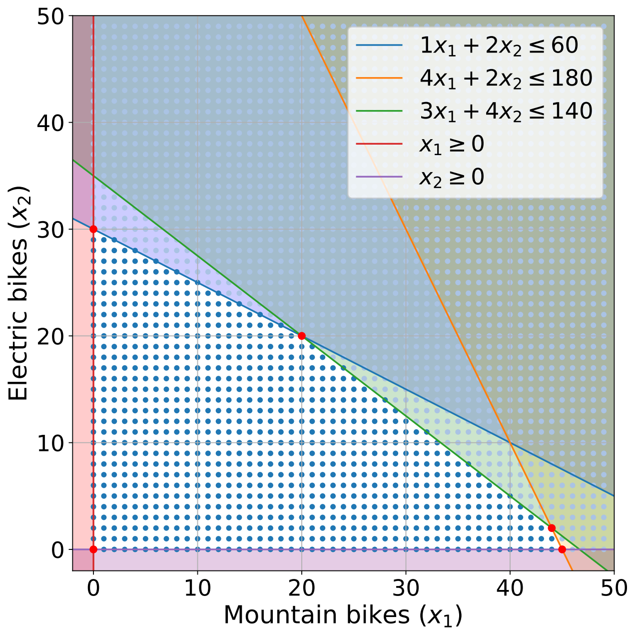

Figura 10: The blank area represents the feasible region of the linear programming problem in Section 3.2, while the blue dots represent the feasible region in its integer counterpart, namely the variables in the bicycle problem must be integer numbers (integer linear programming problem).

overview, we do not have to settle for a local optimum of the non-convex region, but there are several analysis tools that allow us to find the global optimum of the whole region x^*. By having this overview we could save costs and increase the efficiency of processes where the market is unable to do so because of the way it works.

Mixed integer programming

The problem of integer variables has a solution known as mixed integer programming (MIP). This is a variation of linear programming with the addition of the restriction that certain variables must be integer. The restriction makes the problem considerably more difficult to the point of requiring other algorithms such as the "branch and cut" method. The reader interested in this type of problems can consult [43]. An example of a MIP problem is shown in figure 10.

Artificial Intelligence (AI)

Another proposal to deal with non-linearities is transforming the technological matrix A such that the coefficients are no longer a scalar a_{ij}, but a function $f_{ij}(x_j)$able to model scale economies instead [44]. As a result, the technological matrix $F(\mathbf{x})$ changes depending on the amount of each good j to produce.

$$(I - A)\mathbf{x} = \mathbf{d} \quad \longrightarrow \quad (I - F(\mathbf{x}))\mathbf{x} = \mathbf{d}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \longrightarrow \quad F(\mathbf{x}) = \begin{bmatrix} f_{11}(x_1) & f_{12}(x_1) & \cdots & f_{1n}(x_1) \\ f_{21}(x_2) & f_{22}(x_2) & \cdots & f_{2n}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(x_n) & f_{n2}(x_n) & \cdots & f_{nn}(x_n) \end{bmatrix}$$

For this proposal to be feasible for large-scale economic planning, each production unit must be able to accurately model the amount of inputs required to produced each good as a function of the quantity to be produced, thereby making it necessary to generate a large amount of mathematical models. Thanks to recent advances in data science and artificial intelligence, such a process could be, up to a certain extent, automatically done, since the functions f_{ij} could be directly "learnt" based on real production data from each of the production units, as proposed by Spyridon Samothrakis in [45].

4 Computational complexity

In the previous section, we have seen how useful linear optimization can be to rationally and efficiently organize an economy without money. However, L. von Mises argued that this was impossible because of the complexity of the calculations involved [46, 47].[46] The purpose of this section is, on the one hand, to define precisely what is meant by "complexity of the computations involved"–what we will later call the *computational complexity* associated with a certain algorithm–and, on the other hand, to show the computational complexity of certain algorithms.

When Mises formulated his famous critique of economic planning, the word *computer* was associated with a job and not with the object used today to connect to the Internet, capable of performing billions of arithmetic operations per second. In the case of the USSR, a computer could be imagined as a party official sitting in some Gosplan office performing additions and subtractions at full speed, which entailed two major limitations: the size of the problem to be solved was very limited and calculation errors were the norm. However, these limitations have been overcome by today's computers, which can perform the same operations without failure and at a speed unimaginable for a human being. In this latter case, performing economic calculations rationally will only be possible if enough algorithmic and computing resources are available to solve linear programming problems in time [48].

4.1 The Concept of Complexity

Computational complexity analysis is a branch of algorithmics that precisely tries to answer the question of whether a given mathematical problem can be solved under current technological conditions. The complexity of an algorithm is given by the ratio between the number of simple arithmetic operations and the number of variables in the problem. For example, counting the number of characters in a sentence (e.g., the sentence "Let's plan!", which has 11 characters), we only need to iterate over each character, beginning with the first one, and with each new iteration add 1 to a counter originally set to 0. Each of these two operations, iterating and adding, are repeated for each new character, and hence we say that this function has a complexity n, where n is the number of characters in the phrase. To express the complexity of the algorithm we use what is called the big-O notation, which expresses that the number of operations

[46] In his own terms: "It has been argued that in a socialist economy it would be possible to solve the problem of economic calculation by implementing the equations, considering the description given by mathematical economics of the conditions of economic equilibrium. [...] [However] Hayek (1935) estimates the order of magnitude of the number of equations and calculations necessary as hundreds of thousands. [...] It is clear that the multiplicity of data, and the corresponding establishment of the equations, is an arduous task beyond central planning. The practical impossibility of carrying out the proposals related to this or any similar solution is certainly undisputed" [46].

of the algorithm is *proportional* to, but not exactly, the number of variables in our problem n. Therefore, we say that the complexity of counting characters in a sentence is $O(n)$.

But, why not getting the exact number of operations instead? This can be more complex than it seems at first glance, as it depends on many different factors, such as the processor on which the algorithm will be executed, the programming language used to implement it, how efficient the implementation itself is, etc. Therefore, the big-O notation provides a good approximation under which to group algorithms according to their complexity without taking all these factors into account.

We say that an algorithm is "feasible" or "fast" enough to be executed on current computers if it belongs to the class of algorithms with complexity P. The P denotes that these algorithms can be solved in (P)olinomial time; that is, their execution time is bounded by a polynomial expression of the type $O(n^k)$ where k is a positive constant. Within this group, we say that problems of linear complexity $O(n)$ (i.e. $k = 1$) have the lowest degree of complexity, closely followed by logarithmic complexity $O(n\log(n))$. In the next complexity level are the problems with polynomial complexity of order greater than 1 (i.e. $k > 1$), for example $O(n^2)$ or $O(n^3)$ for $k = 2$ and $k = 3$, respectively. However, these problems can still be executed fast enough in most computers nowadays, provided they have enough computational resources.

On the other hand, there are the non-deterministic polynomial time problems (known as *NP* problems), which have complexity $O(e^n)$.[47] We say that this last group of problems is computationally intractable, since the number of operations grows exponentially with the number of variables, which would quickly exhaust even the fastest computer. At first, this division between simple and complex problems may seem a bit arbitrary, especially when we do not understand exactly what is meant by a logarithm or the exponential function. However, the reason of the division becomes crystal clear when comparing the different compelxity functions in the same graph (see Figure 11).

[47] Euler's number e is approximately 2,718.

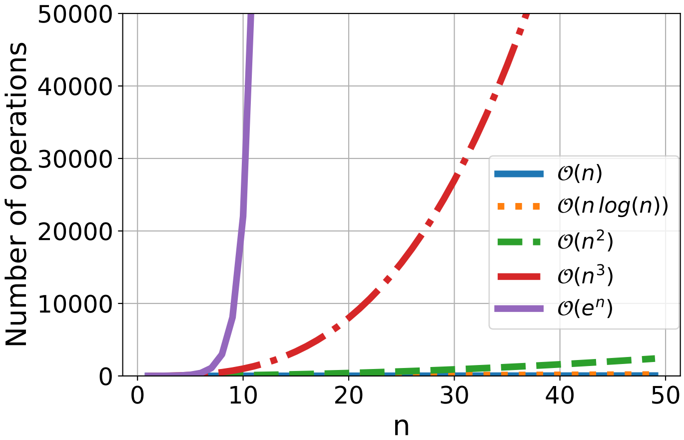

Figura 11: Evolution of the computational complexity as a function of n.

We can better illustrate the differences between P-type and NP-type problems with an example. The complexity of multiplying two prime numbers has low complexity regardless of the size of those numbers (e.g., $1180 \cdot 1233 = 1,454,940$ and $1 \cdot 2 = 2$ involve a "single multiplication" each).[48] However, if we propose the inverse problem, namely finding the two prime numbers that multiplying them yields a certain number, it may seem like a simple problem on a small scale (e.g., $15 = 3 \cdot 5$), but it becomes extremely complicated as the numbers become slightly larger (e.g., what are the two prime numbers that yield $8,003$ when multiplied?[49] At first glance, the problem of finding prime numbers may seem uninteresting, but cryptography is based on the difficulty of solving such problems and thanks to it we can, for example, prevent our data from being visible to anyone when transmitting them over a WiFi access point.

When looking at Figure 11, it seems obvious that the linear, logarithmic and polynomial $O(n^2)$ cases have much lower complexity than the rest. At the other extreme is exponential complexity, which becomes almost-vertical extremely rapidly. Actually, the exponential concept is often counter-intuitive because we humans are not used with dealing with such large magnitudes in a day-to-day basis. Perhaps an example will give the reader a better idea: the number of atoms in the known universe is estimated to be, in the smallest case, 10^{80} (a 1 followed by 80 zeros).[50] The exponential function would exceed this value for $n = 185$! In other words, if n were the number of items in an economy, exponential complexity algorithms could not even be used to plan a household economy. A cunning reader may have noticed that, at first glance, the polynomial complexity $O(n^3)$ would behave similarly, only it grows at a slower rate. Nothing could be further from the truth! The key to understanding this issue is

[48] Integer multiplication is, as of 2019, $O(n \log n)$. See Multiplication algorithms.

[49] Spoiler alert: $8,003 = 53 \cdot 151$

[50] https://en.wikipedia.org/wiki/Observable_universe#Matter_content%E2%80%94number_of_atoms

at the point $n = 10$, for which the exponential complexity is ten times larger than the polynomial $O(n^3)$. This difference will grow *exponentially* with every increase of n and thus they are not similar at all. The latter type of problems will require more time to solve, but today's computers can still do it at such a speed that we can say they are feasible to run even for large values of n.

Having presented the different complexity types, how the complexity of a given algorithm can be obtained remains an open question. This is a fundamental exercise to understand what the computational complexity really is. Section 2 has introduced the inverse of the Leontief matrix as a fundamental element for economic planning. Therefore, the complexity involved in the inversion of a square matrix is an excellent exercise for understanding the computational complexity in the economic planning domain. Although it is not the most efficient method for inverting a matrix, the Gauss-Jordan algorithm[51] is excellent for illustration purposes and hence will be introduced in the following section, together with a derivation of its complexity.

4.2 Matrix Inversion: Gauss-Jordan Method

Let's assume an invertible, square matrix A. The Gauss-Jordan method consists of finding the "elementary" operations[52] that convert the matrix A into the identity matrix I. Once they are found, these operations must be applied to the identity matrix I in the exact same order to obtain the inverse matrix A^{-1}.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

In the following, we present, step by step, the required operations to convert A into I[53]:

1. Divide row 1 by $a_{1,1}$. Operations performed: 3 divisions.
2. Subtract row 1 multiplied by $a_{2,1}$ from row 2. Operations performed: 3 multiplications and 3 subtractions.
3. Subtract row 1 multiplied by $a_{3,1}$ from row 3. Operations performed: 3 multiplications and 3 subtractions.

[51] https://es.wikipedia.org/wiki/Eliminaci%C3%B3n_de_Gauss-Jordan

[52] These operations consist of adding multiples of another row to a row, multiplying a row by a non-zero scalar or changing rows of order. Readers interested in finding out more about linear algebra can consult, for example, [49], [50]. There are hundreds of good textbooks that extensively deal with the subject.

[53] We can assume that $a_{1,1} \neq 0$ since as A admits inverse matrix no column of A can have all null entries, therefore, by swapping the order of the rows we can always guarantee it.

After these first three steps, the matrix A has become the matrix A', whose first column is identical to the first column of I.

$$A' = \begin{bmatrix} 1 & a'_{1,2} & a'_{1,3} \\ 0 & a'_{2,2} & a'_{2,3} \\ 0 & a'_{3,2} & a'_{3,3} \end{bmatrix}$$

4. Divide row 2 by $a'_{2,2}$.[54] Operations performed: 2 divisions. (remember that $\frac{0}{a'_{2,2}} = 0$).

5. Subtract row 2 multiplied by $a'_{3,2}$ from row 3. Operations performed: 2 multiplications and 2 subtractions.

$$A' = \begin{bmatrix} 1 & a'_{1,2} & a'_{1,3} \\ 0 & 1 & a''_{2,3} \\ 0 & 0 & a''_{3,3} \end{bmatrix}$$

6. Divide row 3 by $a''_{3,3}$.[55] Operations performed: 1 division.

$$A'' = \begin{bmatrix} 1 & a'_{1,2} & a'_{1,3} \\ 0 & 1 & a''_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$$

At this point, the diagonal of the matrix already has all its components at 1. Now it only remains to proceed in the same way, but in the reverse direction.

7. Subtract row 3 multiplied by $a''_{2,3}$ from row 2. Operations performed: 1 multiplication and 1 subtraction.

8. Subtract row 3 multiplied by $a'_{1,3}$ from row 1. Operations performed: 1 multiplication and 1 subtraction.

9. Subtract row 2 multiplied by $a'_{1,2}$ from row 1. Operations performed: 1 multiplication and 1 subtraction.

[54] We can assume, again, that $a'_{2,2} \neq 0$ since no column of the obtained submatrix can have all null entries.

[55] We can assume, again, that $a'_{3,3} \neq 0$ since no column of the obtained submatrix can have all null entries.

$$A''' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

After all these operations, $A''' = I$. In total, this process required a total of 28 simple arithmetic operations (6 divisions, 11 multiplications and 11 subtractions).[56] However, this is only for the case where A is a 3×3 matrix. We must generalize the computation of the operations for an arbitrary $n \times n$ matrix.

4.3 Matrix Inversion: Complexity

The Gauss-Jordan method could be viewed as an iterative method in which we operate on smaller and smaller matrices. Given the matrix $n \times n$ shown below and starting from the first *pivot* (the pivots are the elements in the diagonal of the matrix and they must become 1), the objective is to make 0 all the elements in its same row and column, and then perform the same operation with the $(n-1) \times (n-1)$ matrix.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & & & & \\ a_{3,1} & & & & \\ & & (n-1) \times (n-1) & & \\ \vdots & & & & \\ a_{n,1} & & & & \end{bmatrix}$$

As before, we assume that A is invertible[57] and consequently we can assume that $a_{1,1} \neq 0$ otherwise we could swap the order of the rows to guarantee it.

In order to make the first pivot 1, it is necessary to perform n divisions (divide the first row by $a_{1,1}$). For the first element in each row i to become 0, it will be necessary to perform n multiplications (multiply the first row by $a_{i,1}$) and n subtractions (subtract the n elements of row i with those of row 1). This operation must be repeated for each of the $n-1$ rows. It only remains to make the last $n-1$ elements in the first row 0, which can be done by multiplying the pivot in column j by $a_{1,j}$ and subtracting the result from that position in the first row. Therefore, this operation requires $n-1$ multiplications and $n-1$ subtractions. Finally, to obtain the total number of operations

[56] The row changes in the cases where the 'pivot' was null do not imply any change in the computational complexity of the algorithm.

[57] Otherwise the goal of the algorithm could not be achieved.

$N(n)$, we must add the operations required for the $(n-1)\times(n-1)$ matrix to the number of operations obtained above,[58] resulting in the equation (27).

$$N(n) = \underbrace{N(n-1)}_{\text{Recursivity}} + \underbrace{n}_{\substack{\text{Divisions to}\\ \text{make the pivot 1}}} + \underbrace{2\cdot n\cdot(n-1)}_{\substack{\text{Multiplications and}\\ \text{subtractions to make}\\ \text{the row 0}}} + \underbrace{2\cdot(n-1)}_{\substack{\text{Multiplications and}\\ \text{subtractions to make}\\ \text{the row 0}}} \tag{27}$$

The lat two elements in the summation in equation (27) can be combined into a single term, giving rise to the expression $N(n) = N(n-1) + n + 2\cdot(n^2-1)$. At this point, all that remains is to continue the iterative process:

$$N(n-1) = N(n-2) + (n-1) + 2\cdot((n-1)^2 - 1)\ . \tag{28}$$

In the equation above, we have simply started from $N(n)$ and substituted n for $n-1$, thus applying the concept of *recursivity*. In order to obtain the total number of operations, we will have to continue for $n-2$, $n-3$, etc. until we reach $N(1) = 1$, because the smallest matrix we will find is the 1×1 matrix corresponding to the last pivot, which only needs one division to become 1. Expressed in mathematical notation, the complexity of the Gauss-Jordan method will be given by:

$$C = \sum_{k=1}^{n}[k + 2\cdot(k^2-1)] = \sum_{k=1}^{n} k + 2\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} 1\ . \tag{29}$$

How do we go from C to $O(n^3)$? Luckily, the equation (29) can be simplified using a few mathematical tricks. The first of these is fairly obvious: adding n times 1 equals n. On the other hand, the first term of C is what is known in mathematics as an *arithmetic progression*, which can be simplified in a very original way:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n\ .$$

It can also be rewritten in the reverse direction:

$$\sum_{k=1}^{n} k = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1\ .$$

Adding both expressions we obtain that $2\sum_{k=1}^{n} k = n(n+1)$ since there are n sums of the form $j + (n-j+1) = n+1$. The expression for $\sum_{k=1}^{n} k^2$ will be obtained from the previously deduced formula. On the one hand we have that

$$\sum_{k=1}^{n}[(k+1)^3 - k^3] = (n+1)^3 - 1 \tag{30}$$

[58] When moving to successive pivots, we must always ensure that the pivot is non-zero, which can be done by swapping rows son long the matrix is invertible. We assume that row swapping does not lead to changes in the computational complexity.

and on the other hand it is obtained that

$$\sum_{k=1}^{n}[(k+1)^3 - k^3] = \sum_{k=1}^{n}[3k^2 + 3k + 1] = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n. \tag{31}$$

Putting together (30) and (31) we obtain,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\ . \tag{32}$$

At this point, the complexity of the Gauss-Jordan method can be rewritten as follows:

$$C = \frac{n(n+1)}{2} + 2\cdot\frac{n(n+1)(2n+1)}{6} - 2n = \frac{4n^3 + 9n^2 - 7n}{6}\ . \tag{33}$$

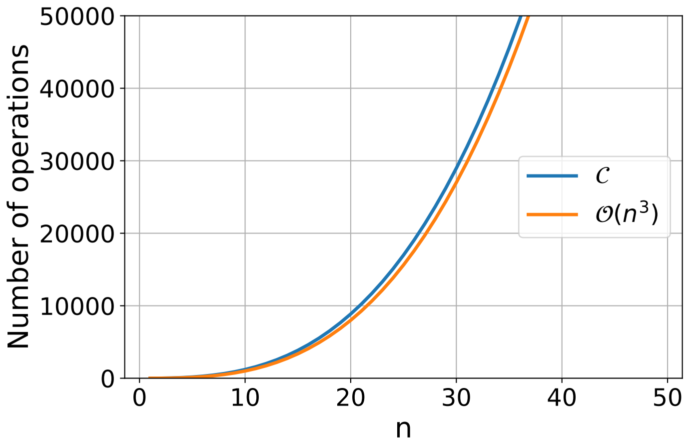

Figura 12: Complexity comparison between $O(n^3)$ and the exact number of operations of the Gauss-Jordan method.

We recommend that the reader evaluates the equation (33) at $n = 3$ and checks that they indeed obtain the same result calculated for the matrix 3×3 in section refsec:method-gauss-jordan.

At this point, only one step remains to obtain the complexity in large O notation. For n large enough, the element with the largest exponent will dominate in the obtained expression the total complexity (see Figure 12). Therefore, the computational complexity of the Gauss-Jordan Method is $O(n^3)$.

4.4 Complexity and economic planning

Having understood the concept of computational complexity, we are now ready to ask: could the simplex method be used to plan a modern economy? Different algorithms for the simplex method have been proposed over the last decades, all of them with polynomial complexity, as shown in the table 4. The possibility (or impossibility) of using such algorithms to plan an economy will depend on the final value of n, the number of items to be produced.

We can assume that, in a modern economy with high degrees of specialization, the number of products is proportional to (but probably less than) the number of inhabitants, since the production of certain complex products (e.g., vehicles, houses, computers, etc.) requires the work of more than one person. As a result, an upper limit for the parameter n would be, being generous, in the order of 10^9 for economies as large as India's or China's.

Surprisingly enough, supercomputers using parallel computing on thousands of processors at the same time were already able to solve problems of such a magnitude in less than an hour since 2006 [51] using parallel computations of thousands of processors.[59] For more modest problems with n on the order of 10^6, the 4-core processors of most desktop computers would be sufficient [53].

Author	**Year**	**Complexity**
Khachiyan	1979	$O(n^6)$
Karmarkar	1984	$O(n^{3,5})$
Renegar	1988	$O(n^{2,873})$
Vaidya	1989	$O(n^{2,5})$
Lee y Sidford	2015	$O(n^{2,5})$
Cohen, Lee y Song	2020	$O(n^{2,373})$

Cuadro 4: Computational complexity of the different algorithms proposed for the simplex method [54].

As we can see, the *economic calculation problem* in socialism, at least from a technical point of view, is not unsolvable at all. Both computational complexity analyses and empirical results show that handling optimization problems of the size of today's largest economies is clearly feasible. However, it should be noted that the problem of economic calculation is not posed only in *technical* terms, as we have considered so far, but would also have a purely *economic* character, related to human evaluation and decision on the ends and means of productive activity [48, 55]. It is precisely on this

[59]The latest research in this area has been carried out by Tomas Härdin in [52], where he analyzes the capabilities of current supercomputers to solve linear problems with sparse matrices (i.e., with most inputs at 0), since it is expected that the inputs to each of the industries is a small subset of all available outputs in an economy, giving rise to sparse technology matrices.

last point where part of the current Austrian critique of the planned economy is focused and that is why, at Cibcom, we treat this issue as one of the keys to put socialist planning back in the game.

5 Conclusions

Most of the problems economic planning faced in the so-called "countries of real socialism" can be solved today thanks to the use of certain modern mathematical and computational tools. Despite how intricate they may seem at first sight, it is possible to take a pedagogical approach to them, allowing us to appreciate their practical implications. The economic problem, summarized in the introduction as "Logistics, Development, Feasibility", receives a clear, coherent, and solid answer from the cyber-communist program.

When it comes to logistics, the intention is to manage logistics issues more precisely and efficiently than capitalist markets and their mechanisms: disciplinary competition between independent capitals and monetary profitability as a generalized one-dimensional incentive. It is proposed that this is done by avoiding the aforementioned vices of the early Soviet economy, namely, planning the number of goods of each type by trying to anticipate or predict citizens' needs. What lessons can we draw from this? What are the demands that derive from our development?

1. The cost of goods must be *proportional to the social costs* and organically consider the scarcity of natural resources.
2. Planning must be established in relation to the needs effectively expressed in a plurality of social instances, forming a *feedback system between the Administration and its different ramifications* at different scales.[60]
3. An *alternative universal accounting system* is necessary to be able to represent or homogenize different bundles of goods with a single common unit. If we are not able to use a non-monetary one, the use of money will prevail, with all the problems that this entails.

Starting with the latter, cyber-communism contemplates the so-called *integrated labor costs* as the only magnitude capable of assuming this function, which entails two major benefits. The first and most obvious one is that, incorporated into an *input/output methodology*, they lay the foundations for the elaboration of highly detailed economic plans. In order to be able to calculate the ILC, it is necessary to know how may units of each type of good are needed to produce a unit of any other good. It is thanks to this that we can carry out our basic planning strategy: to determine from the *technology matrix* the number of units of each type of good that must be produced to satisfy a *constantly updated final demand*. As we have seen, this requires only the calculation

[60] As we will try to explain in the future, from cyber-communism, "[direct] democracy is important, not [only] for moral reasons, but for being able to capture, in detail and in as much quantity as possible, information about demand, as well as to capture good ideas that are currently ignored by the market system" [56].

of the inverse Leontief matrix.

The second benefit is something we did not cover in any great detail: ILCs allow the establishment of an alternative compensation system to wages as we know them, based on *labor bonds or credits*. This makes it possible to purchase certain types of consumer goods in public stores without the problem of persistent imbalances between effective supply and demand (described in Appendix A). For more information on this proposal, we recommend reading [10].

In any case, the formula for the success of the cyber-communist organization of logistics is the socialization of economic activity, an *administrative centralization* that allows us to know and mobilize the available resources democratically, overcoming the informational opacity constitutive of private property.

In any advanced economy with multiple industries and production techniques for each of the products, it is necessary to employ a resource *distribution function* among the different economic tasks. This is what we called the development problem. In capitalist societies, this function is carried out by the market, with its disastrous periodic crises resulting from competition, and by the entrepreneurial function, with the lack of democratic control of the economy that this implies. Instead, we propose the expropriation of the means of production and their collective management by means of mathematical optimization tools for the execution of economic decisions. Optimization makes it possible to maximize or minimize a certain objective function, such as the amount of work in a society or the CO_2 emitted, taking into account the biophysical limits of the system (amount of raw materials consumed, available working hours, etc.). Using this alternative approach, we aim to achieve greater efficiency, since optimal decisions would be made given a certain technological level, and in turn allow a democratic control over the economy, either through plebiscites or through the population's consumption reflected in demand.

The basic methodology presented to answer these problems is *linear programming*, which consists of optimizing linear functions in a convex polyhedron. As we have seen, this is incredibly versatile. The family of microeconomic and macroeconomic problems that can be formalized using linear programming is extensive.

Nevertheless, as one could expect, the jewel in the crown of the cyber-communist initiative is the promise that all these calculations and operations are computable in a reasonable time and with sufficient approximation. In other words, that it responds to the so-called feasibility problem.

The idea that this is impossible, that the problem of economic calculation cannot be solved democratically, is appreciable in almost all corners of the political spectrum.

From liberals to market socialists to various forms of social democracy, we find people repeating the arguments of the Austrian school. However, the great advances in recent decades in more efficient optimization algorithms and faster computers seem to indicate quite the opposite.

It is well known that linear programming is easily tractable with computers using algorithms such as *simplex*. In addition, a solution to more specific nonlinear problems, such as fixed costs, economies and diseconomies of scale or integer variables, can be found as long as we bound their scale appropriately. For each of these challenges, there are algorithmic approaches that allow their application in order to maximize the benefits of the plan: piecewise linearization, non-convex optimization, mixed integer programming, artificial intelligence, etc.

This article has provided an introduction to computational complexity analysis, which allows us to systematically analyze the complexity, either in the form of computational resources or computational time, of solving a given algorithm. We believe that any self-respecting critique of the problem of economic calculation should start with the tools offered by computational complexity analysis. The studies referenced in this article show that, given the latest state-of-the-art resources, we are already capable of solving optimization problems for planetary scale economies, so the supposed impossibility of economic calculation does not seem to be as such.

For the sake of clarity, this article has oversimplified the development problem, as show in Section 2.2. For a more realistic and detailed anaysis of this issue, specialists such as Cockshott, Dapprich or Härdin would better serve as references. However, we still have a clear political intention: to show, through the clarification of its more technical edges, the vitality of the communist project, and the foundations on which it can be built.

Finally, let us underline something that is not always made explicit: you do not have to be a cybernetic genius to discover the “magic” behind this research program. The vast majority of us will not have to invent any mathematical techniques or readjust the computational complexity of any five-year plan. Our initiative to disseminate it in this depth is intended, first and foremost, to make larger portions of our class aware of its existence, that we *understand* how hopeful our proposal is. This is our message to all of us who feel the shackles of capital: Comrades, there are possibilities for our long-awaited democracy, that seed that germinated slowly for who knows what future harvests, and whose sprouts will not take long to burst the earth!

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About CibCom

CibCom is an interdisciplinary collective of Spanish origin, but with projection to Latin America, dedicated to the research and dissemination of the emerging cyber-communism (acronym of our name): the proposal orchestrated by Víktor Glushkov and Stafford Beer in the second half of the 20th century, and developed by Paul Cockshott and Allin Cottrell in the last decades. Our ranks are made up of students and specialists from the worlds of mathematics, physics, computer and telecommunications engineering, as well as philosophy, economics and sociology, who, from a Marxist paradigm, insist on the need to overcome the market economy.

We aim to revitalise revolutionary traditions after decades of retreat and retreat. We believe that for the labour movement to regain its transformative potential it must reassert its post-capitalist projection. For this reason, we are mainly dedicated to structuring a solid and coherent theoretical body on what we consider to be the main institutional device of a socialist political programme: the democratic planning of the economy. We must once again denaturalise private property. Workers' power must be able to be realised.

Appendices

A Why should consumer goods be taxed on the basis of integrated labour costs?

In this section we reason why the persistent imbalances between supply and demand in the USSR were no coincidence, but inevitable, due to the lack of proportionality among taxation, effective demand and integrated labor costs (ILC).

K. Marx considered [57] that economic planning can have no other basis than the flows of labor time between the different branches of production: "Once collective production is assumed, the determination of time, as is obvious, becomes essential [...] Economy of time: this is what every economy finally boils down to. [...] Economy of time and planned distribution of labor time among the various branches of production are always the first economic law based on collective production." Consequently, let us begin by making explicit the intersectoral relations that represent flows of labor time. Let us aggregate the economy into three sectors:

(I) The production of means of production.

(II) The production of means of individual consumption that are purchased in public stores against labor certificates or other remuneration system.

(III) The provision of public services, i.e., individuals do not use labor certificates or other form of social retribution for work performed to acquire services such as hospitals and schools.

We will denote the final product (in labor hours) of Sector (I) as production goods (includes wood, steel, cement, ...etc) and denote by m, and denote by M_1, M_2, M_3 the total of production goods used in Sector (I), (II) and (III), respectively. We know that the production goods wear out, let us assume for simplicity that a $delta$ fraction of them wear out each year. Then, in order to maintain each sector at idle state we need δM_1,δM_2, δM_3 labor hours per year, respectively. If m_g is the final annual growth of m, we have

$$m_g = m - (\delta M_1 + \delta M_2 + \delta M_3) \ . \tag{34}$$

Suppose that the total number of people working (denoted by P) is divided into P_1, P_2, P_3 people working respectively in Sector (I), (II) and (III). To make this section as general as possible, we can assume any system of remuneration which is uniform and ubiquitous in which the rest of costs are expressed and with which to acquire the goods of Sector (II), as was the ruble in the USSR, the monetary unit in the USSR.[61] Once the remuneration system has been fixed, all labor flows must be expressed in this common unit. Otherwise, there is no way of coordinating the different sectors of the economy, i.e., it becomes the accounting unit on which economic planning is based.

For the sake of simplicity, let us assume that all working days are equal. Let us denote by w the salary of each worker[62] in the chosen remuneration system. We will call c the total costs expressed in the chosen accounting unit of m. Let us denote by C_1, C_2, C_3 the respective accounting costs of each sector. Since each sector has as expenses the production goods worn out and the salaries of the people working there, we have the following equations

$$C_1 = c\delta M_1 + wP_1 \ , \tag{35}$$

$$C_2 = c\delta M_2 + wP_2 \ , \tag{36}$$

$$C_3 = c\delta M_3 + wP_3 \ . \tag{37}$$

We denote by b the final product of Sector (II) (in hours worked) and by p the number of accounting units that b represents. For simplicity, we assume that people are not able to accumulate accounting units at the end of each year, in other words, they do not save. If t is the income tax rate, we have

$$pb = w(P_1 + P_2 + P_3)(1 - t) \ . \tag{38}$$

Up to this point, c was a variable completely independent of the rest. However, the correct way to define c is by dividing the accounting costs of Sector (I) by the number of labor hours that the final product of the same sector embodies, formally:

$$c = C_1/m. \tag{39}$$

The net accumulation of new production goods and the accounting costs of Sector (III) is financed via the total collected taxes and the net "profits" of Sector (II)[63], i.e., we have

$$cm_g + C_3 = tw(P_1 + P_2 + P_3) + pb - C_2 \ . \tag{40}$$

As a result, we have 8 free variables $m_g, c, w, t, p, C_1, C_2, C_3$ and 7 equations that constrain our choices. If government revenues depend exclusively on taxes, then senatorial prices must correspond to integrated labor costs. Conversely, if taxes collected by the

[61] We propose to abolish money and adopt labor certificates as the unit of remuneration, but we refer to the ruble here in order not to lose generality.

[62] All working hours are remunerated in the same way.

[63] In capitalism this is not true since the personal consumption of the capitalists must be added

government are not sufficient to finance public services (Sector III), then prices of consumer goods (Sector II) must be higher than integrated labor costs, leading to an inflationary effect.

So far we have only discussed the inter-sectoral relationships that have to be verified in any economy, now let us look at the intra-sectoral constraints. Even assuming that the population remains constant, individual consumption is highly variable. Must the prices of each of the products in Sector (II) correspond to their respective integrated labor costs excluding, at most, fluctuations arising from changes in demand patterns? The answer is yes, otherwise the adjustments that individuals make in their demand patterns would be incompatible with the a priori allocation of labor power for each type of item.

Let us look at this with a simple example. Assume that one group of consumer goods, e.g., tables, are extremely devalued compared to another group of goods, e.g., bottles of wine. Let's say that bottles of wine are priced close to their integrated labor cost while tables are half the price of their integrated labor cost. Consumers could switch part of their consumption from wine bottles to tables. Let's say for example they decide to reduce their wine bottle consumption by the equivalent of 5 million labor hours and spend them instead on tables. Given that table prices are equivalent to half of their integrated labor costs, it would seem that those 5 million labor hours that go toward consuming tables (instead of wine bottles) could be enough to buy 10 million labor hours of tables. However, even if the workers who in the past produced those 5 million labor hours of wine bottles were transferred to table production, that would not be enough to produce the 10 million labor hours demanded in tables. More generally, if prices are not proportional to integrated labor costs and effective demand, changes in consumer patterns,[64] would imply either a demand too large to be satisfied with the size of the labor force, or, in the case of switching from one type of good to an overpriced one, a certain portion of the labor force becoming redundant.

Other problems that could arise from subsidizing products would be, on the one hand, the creation of black markets since, as mentioned above, there would be shortages of products and even hoarding of such products by certain groups, and, on the other hand, there would be a duality in the prices of subsidized goods, since the real price of the subsidized goods would be the real price abroad as opposed to the local price.

One of the problems that the USSR persistently faced was an inevitable consequence of the economic policies implemented. The lesson is very simple to learn: we must

[64] Namely, when they stop spending the equivalent of a certain number of labor hours on one type of good to spend it on another type of good.

be able to accurately calculate the integrated labor costs of all types of goods and distribute consumer goods from public stores against the equivalent of their integrated labor costs in order to successfully coordinate the economy without persistent problems of shortages and overproduction.

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