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THE LABOR THEORY OF VALUE AND THE PROBLEM OF JOINT PRODUCTION

The Failure of Sraffa's Theory and Morishima's Misconception

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Abstract: The labor theory of value has been rejected by Morishima on the grounds that it would be incompatible with joint production, which would create negative labor values. This article starts by recalling the various definitions of joint production, as well as the way they relate to the real world. For Morishima, the labor theory of value is a particular case of Sraffa's theory of production prices; it is recalled that in Sraffa's treatment of joint production the occurrence of negative multipliers and therefore of negative quantities comes from the construction of a standard commodity. Morishima extends this demonstration to labor values in the case of joint production, but the article shows that his example of giving negative labor values is absurd. Finally, using a method initially developed by statisticians to deal with joint production and simple matrix calculations, it is demonstrated that it is perfectly possible to obtain positive labor values in a theoretical but realistic model of joint production.

Key words: labor value; joint production; system of national accounts; input–output accounts

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The labor theory of value is at the heart of Marx's economics. It is revealed in the first part of volume I of *Capital* (Marx [1867] 1887). However, in volume III (Marx 1894), edited by Engels in 1894, well after Marx's death, Marx shows in

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Chapter 9 that the formation of a general rate of profit implies the transformation of the values of commodities into prices of production. Then the equalization of this general rate of profit through competition leads Marx in the following Chapter 10 to introduce the concept of market-prices. As he writes:

> The assumption that the commodities of the various spheres of production are sold at their value merely implies, of course, that their value is the centre of gravity around which their prices fluctuate, and their continual rises and drops tend to equalize. There is also the *market-value* . . . to be distinguished from the individual value of particular commodities produced by different producers. The individual value of some of these commodities will be below their market-value (that is, less labour time is required for their production than expressed in the market value) while that of others will exceed the market-value. (Marx 1894, 133; italics in the original)

On this theoretical basis, it should be clear that the first thing to define and calculate in the theoretical field of Marxist theory is the labor value of commodities, and many authors have shown that it is easy to calculate these labor values. However, a serious criticism of the labor theory of value has been based on the difficulties that this theory is supposed to encounter when it is confronted with the problem of joint production. In this article, we will show that it is possible to correctly solve this problem.

The problem of joint production is rarely touched upon by economic theory, certainly because it is a very tangled one, for a number of reasons that will be explained here. It is at the same time a serious one, judging by the fact that the alleged contradictions raised by joint production are the reason why Morishima, in his book *Marx's Economics: A Dual Theory of Value and Growth* (Morishima 1973) ultimately rejected the labor theory of value.

Indeed, in the final chapter of his book (Morishima 1973, 179–196, Chapter 14), Morishima puts into question the validity of the whole theory of value in the form that he has expounded it so far, and intends rather to revise it. He proposes then to abandon the labor theory of value for a von Neumann model of linear programming where equations are replaced by inequalities, and labor values are replaced by minimum labor requirements.

It is thus theoretically relevant to address the problem of joint production, which is all the more complicated as there are several definitions of joint products, a point which will be examined first, noting that these definitions correspond partly to various levels of description in the real world and partly to different theories of values and prices, which should not be unduly mixed.

1. Joint Production and the Various Types of Joint Products

The whole difficulty of the matter comes from the fact that joint products cannot easily be distinguished from by-products. Things would be simple if everybody accepted a strict definition of a joint product as a product that jointly results, with other products, from the processing of a common input or several inputs, with the proportions of inputs going into each product impossible to distinguish. But in practice, this is not the case, even if we turn to the United Nations' (UN) system of national accounts (SNA), updated in 2014 (United Nations 2014). This system of accounts offers the possibility of using a matrix-form of presentation for the accounts, which implies resorting to input–output tables. The methodology to develop such I–O tables is explained in an associated *Handbook of Input–Output Table Compilation and Analysis* (United Nations 1999), last updated in 1999. This handbook distinguishes, among the "secondary products" resulting from production technology, three distinct types of products:

> (a) Exclusive by-products, or products that are not produced separately anywhere, e.g., molasses linked to the production of sugar, new scrap in metal industry;
>
> (b) Ordinary by-products, or products that are technologically linked to the production of other products but are also produced separately elsewhere as main products. An example of this is hydrogen produced as a by-product in petroleum refining establishments, but also produced separately by other establishments in the chemical industry;
>
> (c) Joint products, or products that are more loosely linked technologically than ordinary by-products. The common costs shared by joint products are more significant in value than is the case for ordinary by-products. One example is milk and meat in the livestock industry which may be produced in a scale that depends on the demand for each product and the ratio of the two products may be varied in response to changing conditions of demand. One of the joint products may be produced separately elsewhere. Joint-products cannot be easily distinguished from by-products. (United Nations 1999, 77)

However, when we leave the area of statistics to turn to a purely theoretical definition, when Sraffa addresses the question of joint production in Chapter 7 of *Production of Commodities by Means of Commodities* his definition is more straightforward, since he writes: "We shall now suppose two of the commodities to be jointly produced by a single industry (or rather *by a single process*, as it will be more appropriate to call it in the present context)" (Sraffa 1960, 51; emphasis added). The definition in this very last sentence is much more precise and in fact reduces the category of joint products to category (a) above, i.e., that of exclusive by-products. This means products which share not only one common input, but all of their inputs, and share them as already noted in proportions that are impossible to distinguish.

In the theoretical world, this makes sense, because if we suppose that in an otherwise common process of production for two different products there is only one input (either a commodity or a quantity of labor) for which the proportion going to each product can be distinguished, then the two corresponding production processes will be different, as little as this difference may be. Translated into mathematical language, it means that it will be possible to have two formally distinct and therefore independent equations: one for each process, and here therefore joint production no longer creates a problem for the determination of individual values, i.e., when we have one equation corresponding to one single process for the two unknown values to be determined, but rather where we can write two distinct equations.

In any case, the linkage made by Sraffa between industry and process is quite appropriate, and also interesting because it draws our attention to the complexity of the real world.

2. The Complexity of the Real World as Regards Production

In the real world, or more precisely in the perception that we have of the real world, there are millions of producers. Some of them are small producers, among whom we can distinguish pure individuals who do not have a single employee and cannot be called capitalists in the usual sense of the word. Some entrepreneurs hire a few employees. There are small and medium enterprises, bigger firms or companies, and some of them, the largest ones, have more than a hundred thousand people on their payroll.

It is first in the agricultural sector, where there are sometimes millions of farmers in a single country—each of them producing but a few agricultural products—that there are quite certainly more producers than products, in spite of the many varieties which exist for each agricultural product. And although production processes for one particular agricultural product may not differ greatly, they often differ from one producer to another, due to particular circumstances such as the differences in land quality.

Outside of agriculture, although the number of producers for each particular product is usually smaller, most producers share the common characteristic of also producing several different products each, some of them being specific to only a few producers, and some being produced by many producers, although they may be sold under different brands.

Moreover, a number of producers may produce the same commodity, but usually each producer will have its own process of production, even though each process is only slightly different from the other producers', if only because the machines used by each producer are not exactly the same, or are older/newer, or are not similarly maintained, which may affect the use of various inputs. A single producer may also use different processes, e.g., in different establishments, workshops, or factories, for the same product. As a consequence, there are also millions of production processes, certainly more than the number of commodities themselves. The opposite case is in fact that of only one single producer for one product, which is the exact definition of a pure monopoly, and is not so frequent!

3. The Statisticians' View of Production and Joint Production

All these various cases or circumstances exist in the real world and make for a very complex picture; therefore, we cannot build up a theory of production if we are to stay at this level of complexity. To reduce this complexity to a manageable theoretical level let us first turn to some degree of abstraction, and at the outset to statisticians. In order to describe the production process in an intelligible way, statisticians use a number of simplifying techniques. First, they compile data using the establishment as the statistical unit, which is a production unit consisting of either "an enterprise, or a part of an enterprise, that is situated in a single location and in which only a single [or non-ancillary] productive activity . . . accounts for most of the value added" (European Commission et al. 2009, 87, paragraph 5.2). However, an establishment may engage "in one or more secondary activities, [which] should be on a small scale compared with the principal activity" (89, paragraph 5.15).

Once statisticians get the data corresponding to all of the establishments, they compile them in order to get the production accounts of "industries." According to the SNA definition, an industry consists of a group of establishments engaged in the same, or similar, kinds of production activity. These activities are classified in categories, according to the United Nations International Standard Industrial Classification (ISIC), which contains 17 major sections, 60 divisions, 169 groups, and 291 industries (with four digits coding). More information on this question can be found in the *Handbook of Input–Output Table Compilation and Analysis* (United Nations 1999, 42).

Similarly, statisticians have to reduce the thousands of products of an actual productive system into a meaningful and manageable number: this is achieved through a statistical unit for the products that is a unit of homogeneous goods and services, such as they appear in the *Central Product Classification* (CPC) (United Nations 1999, 48, Appendix B). This classification is quite exhaustive, with all products being mutually exclusive. The detailed classification of products, which are either outputs of domestic production activities or imports from non-resident sources, consists of ten sections, 69 divisions, 291 groups, 1036 classes, 1787 sub-classes, and can accommodate up to 65,610 categories. The principles for classification used by CPC are the following:

> (a) For transportable goods, categories of products should be based on the physical properties and the intrinsic nature of products, i.e., the raw materials of which they are made, their stage of production, the use they are intended for, the prices at which they are sold, whether or not they can be stored, etc.
>
> (b) Individual goods and services as far as possible should contain only goods and services which are produced by a single industry. (United Nations 1999, 43)

As we can see, there can be therefore in these statistical data an important discrepancy between the number of industries and the much higher number of categories of commodities, which is contrary to our initial finding that in the real world there should be more industries, in the sense of production processes, than products. For statisticians, the matrix in which a column shows for a given industry the amount of each commodity it uses as an input is generally called the "use" matrix, and will have therefore more rows (the commodities) than columns (the industries). It will be a rectangular, commodity-by-industry matrix of dimension ($n \times m$) and of rank n. Similarly, the matrix in which a column shows for a given commodity the amount of it produced by each industry, generally called the "make" matrix, will be a rectangular, industry-by-commodity matrix of dimension ($m \times n$) and of rank n.

Bearing in mind that statisticians as well as theoreticians have a common purpose of using the production accounts or equations to perform statistical or mathematical operations, then their objective is to obtain symmetrical input–output tables, of either product-by-product or industry-by-industry tables. Indeed, such tables will allow statisticians to work with square matrices, which are usually invertible, since only a square matrix can be inverted to obtain what is usually called the Leontief inverse matrix. For statisticians this means that a correspondence between ISIC and CPC systems of classification is needed: this is achieved, but at a price, which is the existence of "secondary products," since their classification contains more products than industries. Indeed, in I–O (input and output) accounting, the primary principle is that each industry is associated with a commodity that is considered as the primary product of that industry, and all other commodities produced by this same industry are considered as secondary products.

For instance, in the US system of national accounts, which is certainly one of the best systems currently available, since the year 1997 I–O tables have incorporated a new classification structure known as the North American Industry Classification System (NAICS). NAICS was updated in 2002, and this corresponding NAICS has 20 major sectors with a total of 1179 industries. The US system and I–O tables are described by Horowitz and Planting in a book published in 2006 and updated in 2009 by the Bureau of Economic Analysis, US Department of Commerce: *Concepts and Methods of Input–Output Accounts*.

In spite of the high number of industries in these accounts, there are nevertheless a number of secondary products, because, with the first principle above as the main criterion, different products may come out of the same industry. To give but one example provided by the CPC, published in the Statistical Papers of the United Nations:

> For example, meat and hides are both produced by slaughterhouses. These products are not listed together in one category or even in the same section of the CPC. Unprocessed hides are considered raw animal materials, and they are classified in section 0 (agriculture, forestry and fishery products), whereas meat is classified in section 2, among food products. (United Nations 2015, 8)

It must thus be clear that most of the secondary products of a given industry have nothing to do with true joint products or by-products.

The presence of these products in the make matrix cannot but have a perturbative influence on the calculations which can be made. The United Nations *Handbook of Input–Output Table Compilation and Analysis* indicates indeed that to adhere to the classification of products it is necessary to separate secondary products or joint products from the main products of an industry. Several methods can be used for the treatment of these secondary products resulting from production technology: the negative transfer method, the aggregation or positive transfer method, and the transfer of outputs and inputs. But none of these is fully satisfactory, and sometimes they can involve negative values, which may arise for purely statistical reasons.

These explanations about the statistical field shed some light on the complexity of compiling data on production activities, and bring us finally into the theoretical world, which is Sraffa's and Morishima's world, and the world we are concerned with. Here there is no particular constraint on the maximum number of industries or commodities that can be dealt with. In the next two sections, after showing why Sraffa's as well as Morishima's theories of joint production are wrong, we shall go back to the labor theory of value, demonstrating that values are not incompatible with joint production, and also that even when there is joint production, there is no need to start with square matrices.

4. Sraffa's Theory of Joint Production

There are two important things to note as regards the way Sraffa deals with joint production, which he starts to do in Chapter 7 of his book (Sraffa 1960, 51–54): the first one regards his definition of joint production, which is right, and the second one concerns the way he tries to solve his system of equations, which is wrong, and has nothing to do with values.

Let us underline first that in the universe of theory we can get rid of absolutely all of the secondary products, because there is no statistical limitation regarding the number of industries that we can accommodate. Therefore an "industry" can be defined much more precisely than in the field of statistics: instead of regrouping establishments where "only a single, or (non-ancillary) productive activity . . . accounts for *most* of the value added" (European Commission et al. 2009, 87, paragraph 5.2; emphasis added), which implies *a contrario* the existence of secondary activities and secondary products, nothing prevents us from deciding that an industry regroups all the establishments where a single productive activity accounts for *all* the value added. In fact, it comes down to considering that at the first level of the theory there is a complete correspondence between each commodity and each industry, which is by the way exactly the same assumption as that made by Sraffa and Morishima before they introduce joint production.

In passing, these observations clearly show that the rather vague notion of industry, in the common sense that it has in the real world,[1] is quite different from the statistical concept of industry, which is itself different from the theoretical concept of industry such as it can be defined by economists. It follows that the concept of industry used by statisticians should not be confused with the concept of industry used by theoreticians. This is an interesting epistemological finding.

As far as we are concerned, in adopting a quite restrictive concept of industry corresponding to a particular technical process, we should also be aware that, since there are more establishments and therefore more producers than industries which by definition regroup them, we always deal (and any theory does so) with averages. Indeed, the techniques and methods of production generally differ—even if it is sometimes very slightly, from one producer to another, for obvious reasons of differences in location, productivity, types of machines, etc.

Therefore, the processes for producing the same commodity are never exactly the same for all producers. If we are at the level of the production of one commodity in the whole economy, it means that the conditions of production and the production process for a given commodity correspond essentially to the average process for this commodity at a given time, with such an average spanning many establishments and many production processes. At this stage, in each industry, production of a particular commodity corresponds therefore to a single (and average) production process.

Before introducing joint production into his theory, and as we already indicated, Sraffa is thus right in deciding that the elementary unit of production is an industry identified with a single process of production. The question of the validity of the labor theory of value in the case where several techniques are used to produce the same commodity has been raised, but Toker has shown quite convincingly, in a note on the "'negative' quantities of embodied labor," published in *The Economic Journal*, that such a situation is not a problem, as long as it is understood that what he calls (following Marx) the "market value"[2] of a commodity "is the weighted average of its individual values, weights being the market shares of the respective techniques" (Toker 1984, 152). From this observation, it should be clear that each industry represents an average process of production, as long as joint production does not come into the picture.

After having rightly identified an industry with a single (and therefore an average) production process, Sraffa then gets rid of the assumption that each commodity is produced by a separate industry, to suppose instead that a separate industry, and therefore a single process, can jointly produce two different commodities. From this, it is clear that joint production as he defines it is identified as the production of exclusive by-products, which have nothing to do with the secondary products that are produced by different processes. In such a case (a single process producing two different commodities) Sraffa explains rightly that there will be a single equation for the determination of two prices, and if the situation is generalized, more prices overall than equations, which is not sufficient to determine these prices.

Sraffa also sees clearly what constitutes the correct solution to overcome this difficulty:

> In these circumstances there will be room for a second parallel process which will produce the two commodities by a different method and, as we shall suppose first, in different proportions. Such a parallel process will not only be possible, it will be necessary if the number of processes is to be brought to equality with the number of commodities so that the prices may be determined. We shall therefore go one step further and assume that in such cases a second process or industry does in fact exist. (Sraffa 1960, 51)

Let us stress the fact that this assumption is a perfectly legitimate one, because in the real world, as we already noted, there are always a number of producers, and as already discussed there is no particular reason why it should not be the case for processes (or industries) producing joint products. There is also no reason for these processes to be strictly identical: on the contrary, there is every reason to think that different producers will use production processes that are different, even though the differences are not very important. It is also quite probable that these producers will not produce the joint products (by-products) exactly in the same proportions.

In other words, since industries are identified with processes and since each one results from the aggregation of a number of producers and processes, there is no theoretical difficulty in doing things the other way around and disaggregating a given industry in as many processes (and industries) as the number of by-products that it produces.

Sraffa is therefore right when he generalizes his position in saying that the same result, i.e., the possibility to determine prices, would be achieved "provided that the number of independent processes in the system was equal to the number of commodities produced" (Sraffa 1960, 52), and by considering "a system of *k* distinct processes each of which turns out, in various proportions, the same *k* products" (52). It follows that "an industry or production-process is consequently characterized, no longer by the commodity which it produces, but by the proportions in which it uses and the proportions in which it produces, the various commodities" (53).

On this basis, Sraffa's joint production equations present themselves as follows:

$$\left.\begin{array}{l} \left(A_1 p_a + B_1 p_b + \ldots + K_1 p_k\right)\left(1+r\right) + Lw = A_{(1)} p_a + B_{(1)} p_b + \ldots + K_{(1)} p_k \\ \left(A_2 p_a + B_2 p_b + \ldots + K_2 p_k\right)\left(1+r\right) + Lw = A_{(2)} p_a + B_{(2)} p_b + \ldots + K_{(2)} p_k \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \left(A_k p_a + Bkp_b + \ldots + K_k p_k\right)\left(1+r\right) + Lw = A_{(k)} p_a + B_{(k)} p_b + \ldots + K_{(k)} p_k \end{array}\right\}. \qquad (1)$$

It is nevertheless at this juncture that problems start to arise. Indeed, to be able to deal with changes in distribution (in *w* and *r*), prices and wages must necessarily be expressed in terms of the standard commodity, with wages being defined as a share of the standard net product, which Sraffa intends to construct in the case of joint production in Chapter 8 of his book (Sraffa 1960, 55–65). In order to do so it is necessary for him to transform the above equations, through the definition of ad hoc multipliers, in such a way that products will appear in the same proportions on the left and right sides of these equations. This is a condition for the definition of *R*, the standard ratio (which is equal to the maximum rate of profits when we have $w = 0$).

However, in this new context, some products may appear on the right side of the equations, as joint products, and not on the left side, as means of productions. Since these products cannot for this reason be part of the standard commodity, they have to be eliminated, which implies the occurrence of negative multipliers and therefore of negative quantities in the standard commodity. This does not seem to bother Sraffa, who writes that in the case of the standard commodity:

> [T]here is fortunately no insuperable difficulty in conceiving as real the negative quantities that are liable to occur among its components; these can be interpreted, by analogy with the accounting concept, as liabilities or debts, while the positive components will be regarded as assets. (Sraffa 1960, 56–57)

This quote shows that here we take another step further toward the realm of nonsense, because wages are defined in terms of the standard commodity, as in fact all prices are, which necessarily have the same dimension as the standard, i.e., as their measurement unit, and wages are nothing more than a share of the net standard product. It means therefore that with this new definition and construction of the standard commodity, in Sraffa's theoretical universe workers would be paid in units of this strange thing, made not of final goods, i.e., consumption goods, but of intermediate commodities, and also partly of debts and liabilities!

In the same vein, also in Chapter 8 of *Production of Commodities by Means of Commodities*, devoted to "the Standard System with Joint Products," Sraffa goes as far as explaining that:

> Thus, a standard commodity which includes both positive and negative quantities can be adopted as money of account without too great a stretch of imagination provided that the unit is conceived as representing, like a share in a company, a fraction of each asset and of each liability, the latter as in the shape of an obligation to deliver without payment certain quantities of particular commodities. (Sraffa 1960, 57)

One must nevertheless acknowledge that such a view corresponds to a very strange conception of money, which in fact has nothing to do with the real world.

When a theory thus leads to strange results, and in particular to totally unrealistic ones, as is the case for Sraffa's theory of joint production, one could think that the theory should be abandoned or changed to produce a new or modified theory that would be able to generate more realistic results. But this is not the solution adopted by Sraffa, nor by his followers, who seem to think that since his theory is fine, the nature of wages and money in the real world can be redefined in such a way that it would ultimately fit the theory. This is, however, a strange epistemological attitude, to say the least.

But it is not Morishima's conception, since he decides on the contrary to modify the theory, but in so doing thinks that he can also throw the baby with the bath water, and therefore jettison the labor theory of value. This now leads us to discuss his conception, or rather his misconception, of the labor theory of value, because it is based on the failure of Sraffa's theory to properly integrate joint production.

5. Morishima's Misconception of the Labor Theory of Value

Joint production is an important part of the theory of production prices which produces the bizarre outcomes that we just described, and that leads us away from the real world. One should normally consider that this way of dealing with joint production is thus refuted and then conclude that there is something wrong at least with this part of the theory. Against this background, the next step should be that this particular part of the theory should be abandoned, in order to be reworked or rethought on different bases. But Sraffa could not abandon his general view of joint production, because this was an indispensable element for allowing him to introduce his theory of fixed capital, which he treats in the following chapter of his book. Owing to these shaky bases, it is no surprise that we could show that Sraffa's theory of fixed capital was also deeply flawed, in an article titled "Commodities Do Not Produce Commodities: A Critical Review of Sraffa's Theory of Production and Prices," published in the *Real World Economics Review* (Flamant 2015).

Quite differently from Sraffa, Morishima (1973) does recognize the absurdity of negative quantities, and he is convinced that he can remedy this difficulty by developing an alternative theory, based on von Neumann's model. But what is not understandable is why he considers at the same time that the theoretical flaws of the theory of production prices invalidate the labor theory of value, which has not at all the same conceptual background, in particular as regard to its standard, which is a quantity of labor, with the dimension of time, and therefore has nothing to do with any kind of composite commodity! It must be noted that the word "dimension" has here the same meaning as in the theory of dimensional analysis, which is a part of physics.

For Morishima, the question of the negative quantities of commodities appearing in Sraffa's theory of joint production is indeed considered as the crux of the matter, and as the basis for his strong criticism of the labor theory of value. Following Sraffa, he starts by considering that negative quantities of commodities appear in the theory of production prices. At the same time, he believes that this theory can define the value of commodities as being made of "embodied labor." For him, it means ipso facto that there are negative labor values! But the values as he defines them, i.e., through the same kind of equations as the ones above in section 4 of this article, are nothing more than production prices when the rate of profits is equal to zero. Here we are no longer in the problem of transformation of values into prices, but on the contrary in the reverse logic of the transformation of prices into values, which is another kettle of fish! This explains why he is wrong on several grounds.

First, it must be recalled that labor values are defined at the level of production, before any sale has taken place, which implies that no profit has been yet obtained, whereas production prices are defined at the level of distribution, because they incorporate not only wages but also a uniform rate of profit, which implies (even if the rate of profit is equal to zero) that prices have been realized by the sale of commodities on the market.

Moreover, as we recalled above, the dimension of prices in this theory is the dimension of their measurement unit, i.e., the standard commodity, which is a composite and heterogeneous commodity. It is indeed a necessary condition for the existence of a linear relationship between wages and the rate of profit. In passing, all these particularities show that to speak of production prices is quite inappropriate, and that it would be much better and accurate to name them distribution prices. It remains that these prices have nothing to do with time, which is a dimension of value.

Morishima wants nevertheless to demonstrate that the labor theory of value, like Sraffa's theory of production prices, fails when it comes to joint production, because he considers that labor values are a particular case of production prices! This is the reason why, at the beginning of the final chapter of his book, he tries to explain the reason for this failure, by giving an example of negative labor values (Morishima 1973, 181–182, Chapter 14). But the big problem which then arises is that his example is totally biased.

To show this, let us present Morishima's example with a system of two matrices: a use matrix of inputs and a make matrix of outputs. This system replaces the simple Leontief model, in which one industry produces only one commodity, and each commodity is produced only by one industry. The Leontief model is considered as a symmetric model, with therefore only one square matrix, which cannot handle joint production, because there is no distinction between commodities and industries.

The make–use model was introduced by the *United Nations in the System of National Accounts* (United Nations 1968) in 1968, and is well explained in a paper published in 2002 by Guo, Lawson, and Planting, from the US Bureau of Economic Analysis: "From Make-Use to Symmetric I–O Tables: An Assessment of Alternative Technology Assumptions." We can therefore insert Morishima's data into such a model, which is done in Table 1 below.

We can see from this table that in Morishima's example there is only one intermediate commodity, named good 1, and one capital good, which can serve for two periods. The new fixed capital good is designated as good 2 and the old fixed capital good is designated as good 3. The fixed capital goods are not used to produce themselves but to produce the intermediate commodity, which can thus be produced by two industries (or processes), 1 and 2, utilizing the new and old fixed capital goods, respectively. Like in Sraffa's treatment of fixed capital, the old capital good is a by-product of process 1 using the new capital good. The industry (or process) which produces the new fixed capital good is called process 3.

Table 1. Morishima's Example of Joint Production Presented in a Make–Use Table

	Commodities outputs			*Industries (processes)*			*Total output*
	1	*2*	*3*	*1*	*2*	*3*	
Commodities inputs							
1. Intermediate good				0.7	0.9	0.9	2.0
2. New fixed capital good				0.5	0	0	1
3. Old fixed capital good				0	0.5	0	0.5
Industries (processes)							
1. Using new capital good	1.0	0	0.5				
2. Using old capital good	1.0	0	0				
3. Using no fixed capital	0	1	0				
				Labor input			
Value added	–0.5	0.5	0	1	1	1	
Total input	2.5	0.5	0.5				

Input–output coefficients are given in the table above, where Morishima assumes that process 2 requires a greater amount of the circulating capital good (i.e., 0.9) than process 1 (i.e., 0.7), because the former uses the old fixed capital good and the latter the new one. Then the so-called "values" are calculated from the following system of equations, each corresponding to an industry, where inputs are on the left side, and outputs on the right side:

$$\left.\begin{array}{l} 0.7\lambda_1 + 0.5\lambda_2 + \ldots\ldots + 1 = \lambda_1 + \ldots + 0.5\lambda_3 \\ 0.9\lambda_1 + \ldots\ldots + 0.5\lambda_3 + 1 = \lambda_1 + \ldots + \ldots \\ 0.9\lambda_1 + \ldots\ldots + \ldots\ldots + 1 = \ldots + \lambda_2 + \ldots\ldots \end{array}\right\}. \tag{2}$$

Solving these equations, Morishima obtains negative values: $\lambda_1 = –50$, $\lambda_2 = –44$, $\lambda_3 = –12$. But are his ad hoc system and its results significant? Obviously, the answer is no, because its assumptions are incompatible with common sense, i.e., the sustainability and therefore the mere existence of the system, and this is so because of two errors:

(a) First the sum of the combined inputs of the intermediate good (i.e., 0.7 + 0.9 + 0.9 = 2.5) appearing in the first row of the table, is larger than the production of the same good (i.e., 2), which is incompatible with a self-replacing state;

(b) Second, because the treatment of fixed capital does not correspond either to Sraffa's treatment of fixed capital—however flawed it may be (Sraffa 1960, 76–80)—or to a self-replacing state. Since for Morishima, as well as Sraffa, fixed capital transfers its value to the product, then if there is 1 unit of new fixed capital good 2 produced at each period by process 3 and serving for two periods, which is the case in Morishima's example, then at the beginning of each period and therefore during each period this same unit of new fixed capital good 2 should be used as an input, in whatever process, with half of the value of this new fixed capital good transferred to the product, and the other half transformed into the old fixed capital good 3, as a by-product of the processes where it has been used (in this example, process 1 only). Otherwise, fixed capital does not transfer its value to the product, which is what happens in process 1 (even though it is wrong). This is the reason why there is a net input of 0.5 for the new fixed capital good in the above table, which is not compatible with a self-replacing state. However, the treatment of the half-unit of the old fixed capital good is correct, because at each period its full (residual) value of 0.5 is transferred to the product by industry 2, then without any by-product.

However, Morishima acknowledges that "the case of all goods having negative values is obtained only when the system does not satisfy the conditions of productiveness" (Morishima 1973, 182). In this example, the corresponding input-coefficient matrix is not productive because the system uses more input of intermediate good 1 (2.5 units in total) than its produced quantity (only 2 units). One can therefore wonder why Morishima bothered to give an example that precisely does not meet these conditions, and has therefore nothing to do with the real world.

Then Morishima transforms his example by merging the two first processes producing good 1, and keeping only their average, which gives him the following system of equations (where λ_3 disappears, being on both sides of the total), and which he calls the neo-classical system:

$$\left.\begin{aligned} 0.8\lambda_1 + 0.25\lambda_2 + 1 &= \lambda_1 \\ 0.9\lambda_1 + \ldots\ldots\ldots + 1 &= \lambda_2 \end{aligned}\right\}. \tag{3}$$

He notes that we obtain again the same negative values of $\lambda_1 = -50$, $\lambda_2 = -44$, which is not surprising because the first equation is the average of the former first two. Indeed, this does not prevent the matrix from being non-productive, since there are 1.7 units of good 1 used as input whereas, as it is easy to see, only 1 unit of this good is produced. He concludes from this second system that the productiveness of the neo-classical system is necessary and sufficient for the positivity of the values of the circulating and new fixed capital good.

Finally, from there he develops a last case where the input-coefficient matrix is productive, and which is Morishima's final system of equations:

$$\left.\begin{array}{l} 0.7\lambda_1 + 0.5\lambda_2 + \ldots\ldots + 1 = \lambda_1 + \ldots + 0.5\lambda_3 \\ 0.9\lambda_1 + \ldots\ldots + 0.5\lambda_3 + 1 = \lambda_1 + \ldots + \ldots \\ 0.2\lambda_1 + \ldots\ldots + \ldots\ldots + 0.5 = \ldots + \lambda_2 + \ldots \end{array}\right\}. \tag{4}$$

He notes, however, that despite the fact that we obtain positive values for λ_1 and λ_2, at 7.5 and 2.0, respectively, negative values can still appear, which is the case for the old fixed capital good 3, its value being $\lambda_3 = -0.5$.

In fact, it is so because Morishima does not fully realize the consequences of his own assumption that there is only one fixed capital good, which has a two-period life and thus appears with two ages in the production processes: age 0, when it is produced by process 3 and used by process 1, and age 1 when it is used by process 2. To be sure, this does not prevent from naming the same goods of different ages goods 2 and 3, as he does, but it is not a pure convention, because there is a link between goods 2 and 3. Indeed, on the basis of Sraffa's and Morishima's assumptions regarding fixed capital and the transmission of its value to the product, good 3 is nothing more than good 2 that has lost, because it is 1 year old, a part of its value corresponding to its amortization in the process where it is used.

This forbids absolutely to write the first of the above three equations like Morishima does. In fact, to rewrite it properly, we must first take into account that in process 1 fixed capital good 2 transfers a value corresponding to its amortization, with its residual value, which Morishima seems to set at half of its original value, appearing as a by-product in the form of good 3. We could be tempted to write it as in the following equation:

$$0.7\lambda_1 + 0.5\lambda_2 + \ldots + 1 = \lambda_1 + \ldots + (1 - 0.5)\lambda_2. \tag{5}$$

But we can see that both terms in λ_2 cancel with the same quantity on both sides of the equation, which precludes any transfer of value to the product from the new fixed capital good 2. As a consequence, if there must be any transfer of value to λ_1 (the value of the intermediate good) the equation must rather be written as:

$$0.7\lambda_1 + (1 - 0.5)\lambda_2 + \ldots + 1 = \lambda_1, \tag{6}$$

which gives us:

$$0.7\lambda_1 + \lambda_2 + \ldots + 1 = \lambda_1 + \ldots + 0.5\lambda_2. \tag{7}$$

If we then consider like our author that the residual value of good 2 appears as a by-product, under the form of a distinct good which is fixed capital good 3, it is not 0.5 units, but one full unit of this new good 3, which must necessarily appear on the right side of the equation:

$$0.7\lambda_1 + \lambda_2 + \ldots + 1 = \lambda_1 + \ldots + \lambda_3. \tag{8}$$

The link between the two equations is obviously that $\lambda_3 = (1-0.5)\lambda_2$, assuming a 50% amortization of good 2 in the first period, which is a simple linear amortization rule (meaning that a fixed capital good is amortized in as many equal shares as the number of its periods of use), if we compare it to Sraffa's complicated demonstration, but corresponds to the fact that here there is no rate of profit to complicate the matter. But then in the second equation of this new system, 0.5 λ_3 should in turn be replaced by λ_3, and we would therefore obtain quite a different, but coherent, system, which would be:

$$\left.\begin{array}{l} 0.7\lambda_1 + \lambda_2 + \ldots\ldots\ldots + 1 = \lambda_1 + \ldots + \lambda_3 \\ 0.9\lambda_1 + \ldots + \lambda_3 + \ldots + 1 = \lambda_1 + \ldots + \ldots \\ 0.2\lambda_1 + \ldots\ldots\ldots\ldots + 0.5 = \ldots + \lambda_2 + \ldots \end{array}\right\} \text{a correct system of equations.} \tag{9}$$

However, the two first equations are not independent, because of the amortization rule which links λ_3 to λ_2 (here it is $\lambda_3 = (1-0.5)\lambda_2$), and would make the system overdetermined, with four equations and three unknowns. The only way to avoid this is to add the first two equations, to finally obtain the following system, where we can see that λ_3 has been eliminated:

$$\left.\begin{array}{l} 1.6\lambda_1 + \lambda_2 + \ldots + 2 = 2\lambda_1 \\ 0.2\lambda_1 + \ldots + \ldots + 0.5 = \lambda_2 \end{array}\right\}. \tag{10}$$

The elimination from the first equation of λ_3, i.e., the value of the by-product, confirms that it is not possible to define the value of a fixed capital good by treating it as a by-product while treating it, at the same time, as an intermediate good which transfers its value through an amortization process, something which we demonstrated in our previously cited article, as regards the treatment of fixed capital in Sraffa's system.

In any case, the system of equations (9) above has two positive solutions for values of goods 1 and 2, i.e., $\lambda_1 = 12.5$ and $\lambda_2 = 3$, with the amortization rule giving us the value of good 3, as $\lambda_3 = 1.5$.

From all the foregoing we cannot but infer that Morishima's ad hoc example is meaningless and totally flawed, and proves nothing. But our author does not realize it, and on the contrary uses this supposed inconsistency (coming in fact from his initial assumptions) to derive what he thinks are quite general rules dismissing the labor theory of value. Ultimately such an example justifies his attempt to build his case of replacing the labor value system with a von Neumann model with inequalities, from which are derived minimum labor requirements, rather than labor values. This transforms his system into a theory designed mainly for the choice of the most productive techniques, which has nothing to do with the real world. In this world indeed a number of techniques and processes exist at any point in time for the production of any commodity, and values as social values are necessarily an average of these multiple methods of production.

It seems nevertheless all the more useless to continue discussing at length Morishima's demonstration that it is in fact quite possible to show that joint production is perfectly compatible with the determination and calculation of non-negative labor values, which will be achieved in the next section.

6. Obtaining Positive Values in a Realistic Model of Joint Production

In this section, we will show that labor values can be calculated in a coherent way in the case of joint production, rigorously defined.

The difficulties that we have encountered so far have shown clearly enough that for economists joint production is a true problem and at the same time a tricky phenomenon, which raises a number of question-marks about the best way to deal with it, in particular from a mathematical point of view. Statisticians are generally considered as more familiar with mathematics than many economists, and it should therefore be no surprise that the light at the end of the tunnel of joint production has come from them. Indeed, as early as 1968, they proposed two different methods for transferring secondary outputs and associated inputs by combining the use and supply matrices mathematically in order to build symmetric (and therefore invertible) input–output matrices. A quick presentation of these methods and of the reason for selecting that which appears to be the most appropriate for the resolution of our problem, will allow us to propose a corresponding representation of a joint production system. This will in turn provide a background for a mathematical determination of labor values in such a system.

6.1 Statistical Methods Developed for Dealing with Joint Production

These methods are exposed in the United Nations 1999 *Handbook of Input–Output Table Compilation and Analysis*, already cited. The first method was based on what was called an "industry technology assumption," which "assumes that inputs are consumed in the same proportions by every product produced by a given industry, which means that principal and secondary products are all produced using the same technology, i.e., the same input structure" (United Nations 1999, 86). However, this method was quickly found to break the fundamental economic rule that products with different prices at a given moment (i.e., secondary products, as opposed to the principal product of an industry) must reflect different costs or different technologies. Therefore, the UN statisticians preferred to recommend the use of another assumption, i.e., the "commodity technology assumption," which assumes that the input structure of the technology that produces a given product is the same no matter where (by which industry) it is produced. Although this second method is indeed better adapted to the secondary products that statisticians have to deal with, "it tends to generate negative symmetric input–output tables and requires the make and intermediate matrices of the use table to be squared" (87). This is the reason why this method is not widely used.

However, and as far as we are concerned, joint production is not defined in a way which applies to secondary products that appear in each industry (in the sense given to the word industry by statisticians), because these secondary products are produced with different costs, i.e., with different technologies. On the contrary, our definition of joint production, given above in section 1 of this article, precisely reduces the category of joint products to that of exclusive by-products, i.e., products that have exactly the same input structure. Therefore, the criticism made to the "industry technology assumption" is not relevant in the case of exclusive by-products, and this assumption perfectly fits our definition of joint production.

This explains that we will adopt this "industry technology assumption" for our demonstration regarding the determination of labor values within a system of joint production, and all the more so that this assumption is also attractive for two other reasons: first, it is applicable to the case of rectangular input–output tables, but, and this is the second and most important reason, the method always generates positive symmetric input–output tables, which can be represented by square matrices.

6.2 Joint Production as a Physical Process

Before developing a mathematical solution, we can summarize the new view of production that emerges when we take into account joint production, by drawing a scheme, appearing below in Table 2, and which can be used to describe production as a physical process. Let us stress that this scheme is not a matrix, and that each line represents an industry (as a distinct technical process). The i and j indices refer to commodities and industries, respectively. A rather simple situation would be that only one industry would correspond to each commodity, and thus the distinction made between industries would be the same as the distinction between commodities. These commodities are classified into three categories: intermediate commodities of the first type (listed from 1 to k), which enter directly or indirectly into the production of all the other commodities; intermediate commodities of the second type (listed from $k + 1$ to n), which do not enter into the production of the previous category, but enter only into the production of final goods, and can also enter into their own production; and final goods (listed from $n + 1$ to s).

But in the more complicated situation of joint production, the one-to-one correspondence between commodities and industries (or processes) has to disappear. To be sure the same distinction between various categories of commodities still exists, but we cannot use it in the same way to establish an equivalent distinction between industries. Indeed, we can no longer distinguish between those producing intermediate commodities of the first type, of the second type, and final commodities, because they can produce in their outputs several commodities belonging theoretically to any one of these three categories. If we still can distinguish between industries, it is no more on the basis of their outputs, but only on the basis of the commodities that they use as their inputs: some will use only intermediate commodities of the first type, and others will use both categories of intermediate commodities. Thus, there is no more any reason why the number of industries should coincide with the number of commodities. Therefore, the industries (or processes) using only intermediate commodities of the first type will be listed from 1 to l, the industries using both types of intermediate commodities will be listed from $l+1$ to q and the industries producing final commodities from $q+1$ to t. Commodities will continue to be listed with the same indices as before.

As for the by-products of each industry (or process), most authors having worked on the question, starting with Sraffa, seem to consider that in the area of joint production any industry can produce any kind of commodity, belonging to any of the above categories, or at least such an assumption is implicit in their models, since the question is never really discussed. As far as we are concerned, and on the basis of the very definition of joint products as exclusive by-products with exactly the same structure of inputs, nevertheless, it seems extremely doubtful that an industry should jointly produce as by-products (distinct from secondary products) both intermediate commodities and final commodities, for a first and simple reason that has to do with the differentiation of final commodities.

This means that from one final commodity to the other, and even though two commodities are almost close substitutes, there are differences at the end of the supply chain which imply at the very least some differences in the structure of inputs that have been transformed to produce them. And we already noted that the slightest difference in the input structure of two commodities is sufficient to prevent them from being exclusive by-products. This is true for consumption goods as well as for these other final goods that are machines or more generally fixed capital goods. In fact, by-products are more prone to be found in the upstream part of the production process, i.e., at the stage of production of primary commodities or commodities immediately derived from them, which are also by definition intermediate commodities.

A second and mutually reinforcing reason why by-products of these two different types seem quite improbable is that intermediate commodities are sold by producers only to other producers, and never to consumers. It implies that final goods sold to consumers reach them through distinct and separate distribution and marketing channels that increase or create their differentiation, since they change their cost and input structure. To be sure machines of fixed capital goods are also sold from producers to producers, but as we just noted it would be quite difficult to find out any plausible example of a fixed capital good as a by-product.

All these remarks obviously have a bearing on the possible representations of joint production with exclusive by-products. If we focus first on industries (as processes) whose only inputs are intermediate commodities of the first type (listed from 1 to k), we can reasonably assume that there will be processes (listed from 1 to l) that produce by-products of the same type (listed from 1 to k), and even simultaneously by-products that are intermediate commodities of the second type (listed from $k+1$ to n). But it seems highly improbable that such processes simultaneously produce final goods (listed from $n+1$ to s) as by-products, for the reason just developed, which explains that this possibility will therefore not be retained.

A second category of processes (listed from $l+1$ to q) includes those that use as inputs both types of intermediate commodities, and since by definition intermediate commodities of the second type do not enter into the production of intermediate commodities of the first type, the by-products resulting from these processes will only be intermediate commodities of the second type.

As for final commodities (listed from $n+1$ to s), even though it is difficult to imagine a process which would produce intermediate goods of the first type and simultaneously a consumption good or a fixed capital good as a by-product, one cannot *a contrario* preclude that some particular processes (listed from $q+1$ to t) might produce simultaneously intermediate goods of the second type and such final goods (as an example one can think of fruit-trees providing not only fruits—or final goods, but also some wood—or intermediate goods). Then the corresponding production system would have to be represented as in the scheme appearing in Table 2, where B_{ij} corresponds to the quantity of product i used by industry (or process) j, and D_{ij} to the quantity of product i produced by industry (or process) j.

Table 2 is built on the assumption that in spite of the existence of exclusive by-products, which might potentially imply that any industry could simultaneously produce any kind of goods, be they intermediate or final, there are nevertheless various types of industries, depending on the nature of commodities that they produce. This seems indeed to correspond more closely to the real world. It means that industries noted from 1 to l use only intermediate goods of the first type to produce as exclusive by-products both intermediate goods of the first type, noted from 1 to k, and of the second type, noted from $k+1$ to n. Industries noted from

Table 2. A Schematic Representation of a Joint Production System with By-Products

B_{11}	...	B_{k1}	0	...	0	,	L_1	$\rightarrow$	D_{11}	...	D_{k1}	$D_{k+1,\,1}$	...	D_{n1}	0	...	0
...	...	...	...	...	...	,	...	...	...	...	...	...	...	...	...	...	...
B_{1l}	...	B_{kl}	0	...	0	,	L_l	$\rightarrow$	D_{1l}	...	D_{kl}	$D_{k+1,\,l}$	...	D_{nl}	0	...	0
$B_{1,\,l+1}$	...	$B_{k,\,l+1}$	$B_{k+1,\,l+1}$	...	$B_{n,\,l+1}$	,	L_{l+1}	$\rightarrow$	0	...	0	$D_{k+1,\,l+1}$	...	$D_{n,\,l+1}$	0	...	0
...	...	...	...	...	...	,	...	...	...	...	...	...	...	...	...	...	...
B_{1q}	...	B_{kq}	$B_{k+1,\,q}$	...	B_{nq}	,	L_q	$\rightarrow$	0	...	0	$D_{n,\,q+1}$	...	D_{nq}	0	...	0
$B_{1,\,q+1}$	...	$B_{k,\,q+1}$	$B_{k+1,\,q+1}$	...	$B_{n,\,q+1}$	,	L_{q+1}	$\rightarrow$	0	...	0	$D_{k+1,\,q+1}$	...	$D_{n,\,q+1}$	$D_{n+1,\,q+1}$	...	$D_{s,\,q+1}$
...	...	...	...	...	...	,	...	...	...	...	...	...	...	...	...	...	...
B_{1t}	...	B_{kt}	$B_{k+1,\,t}$	...	B_{nt}	,	L_t	$\rightarrow$	0	...	$D_{k+1,\,t}$	D_{nt}	...	D_{nt}	$D_{n+1,\,t}$	...	D_{st}

$l+1$ to q use both types of basic commodity to produce as exclusive by-products intermediate goods of the second type only, noted from $k+1$ to n. Finally, industries noted from $q+1$ to t use both types of intermediate commodities to produce as exclusive by-products both intermediate goods of the second type, noted from $k+1$ to n and final goods, noted from $n+1$ to s.

It must also be emphasized that by-products, despite the attention devoted to them, are not so frequent in the real world and that when they occur there are rarely more than two by-products produced by the same process. As a consequence, even in the areas (in the above table) where some outputs are deemed potentially to be found, most of the quantities in the cells could well be zero. But this should not jeopardize the mathematical representation of the system.

Coming back to the industry technology assumption, let us recall that its basic assumption is that inputs are consumed in the same proportions by every commodity produced by a given industry, and thus in fact by every process, since each process corresponds to one industry. Let us stress again the fact that if such an assumption cannot be considered acceptable in the case of secondary commodities which are produced with different technologies, i.e., with different actual costs, by the same industry, it is nevertheless perfectly appropriate and acceptable in the case of pure or exclusive by-products, which are produced with exactly the same technology, and therefore the same proportions in inputs and the same costs.

On the particular point of the definition of an industry in a system of joint production, since one process can produce several by-products, and in order to have as many equations as unknowns, we must adopt the same assumption as Sraffa, who on this particular question was quite right: in order for the number of processes to be brought to equality with the number of commodities we must assume that each time an industry produces several by-products it can be decomposed (or disaggregated) in as many industries (or processes) as the number of by-products that it produces. Alternatively, this same equality between the number of processes and by-products can also be achieved in another way, as Sraffa also rightly points out:

> even if the two commodities were jointly produced by only *one* process, provided that they were *used* as means of production to produce a third commodity by two distinct processes; and, more generally, provided that the number of independent processes was equal to the number of commodities produced. (Sraffa 1960, Chapter 7, Paragraph 50, 52; emphasis added)

From what we saw about processes and commodities, we can in any case consider this assumption as quite realistic, apart from the last corollary established by Sraffa, and according to which *the number of independent processes was equal to* *the number of commodities produced*. This equality is indeed a very particular case, which has no reason to exist in the general case and above all in the real world. If we recall that an industry is an intellectual construction, built through the compilation of data from many individual producers, each most often with its own process, each industry represents an average process or method of production. Therefore, it is an assumption which is clearly neither unreasonable nor unrealistic to make: in the real world, there are certainly more processes (or industries) than commodities, which obviously has an important consequence on the mathematical representation of production processes, since it means that matrices will not be square, but rectangular.

6.3 A Mathematical Presentation of Joint Production

The starting point of this mathematical presentation is based on the make–use input–output tables developed by statisticians from the US Bureau of Economic Analysis (Guo, Lawson, and Planting 2002) and derived from the input–output tables of the SNA as established by UN statisticians (United Nations 1999, 88). To begin with the notations, we will keep the indices used so far, and therefore:

s is the number of commodities (decomposed in k, $n-k$ and $s-n$)

t is the number of industries (decomposed in l, $q-l$ and $t-q$)

U, of dimensions ($s \times t$) is the intermediate table of the use table (commodity by industry)

B, of dimensions ($s \times t$) is the use coefficient matrix (commodity by industry)

V, of dimensions ($t \times s$) is the make matrix (industry by commodity) transposed of the supply table M, of dimensions ($s \times t$)—commodity by industry, describing domestic production

D, of dimensions ($t \times s$) is the commodity-output-proportions matrix (industry by commodity)

g_t is the column vector of industry output

q_s is the column vector of commodity output

$\widehat{g}$ is the diagonal matrix of industry output

$\widehat{q}$ is the diagonal matrix of commodity output

With these notations, the scheme of a joint production system as represented above in Table 2 can be represented in Table 3 below, where we have the two matrices U and V as defined above, and where the elements in capital letters represent quantities of the various commodities.

Regarding now the notations, the usual practice in economic theory is to name a_{ij} the production (or input–output) coefficients, which correspond to the share of the total quantity of a product i used by an industry (or process) j.

Since one industry produced only one commodity, and reciprocally, these coefficients corresponded to a commodity-by-commodity square matrix, and we will keep them in this role. But the industry technology assumption, with which we are now working, allows us to start from a more general case of rectangular input–output tables, which are normally expected in the case where there are more industries (l, q, or t) than commodities (k, n, or s). As we shall see below this will not prevent us from generating symmetric input–output tables.

Reserving thus the a_{ij} notation to the particular case of a commodity-by-commodity square matrix, the data upon which we must rely are as usual the methods or processes of production, represented by the share of each commodity's total output that is used as an input by each industry. We will rename these coefficients as b_{ij}, in the case of the use coefficient matrix (commodity by industry) derived from use matrix U in Table 3, and which is therefore named B. This commodity-by-industry, direct requirements matrix B has s rows, corresponding to s commodities, and t columns, corresponding to t industries (or processes), and is therefore a rectangular matrix of dimensions ($s \times t$). It shows (in rows) the share of each commodity's total output that is used as an input by each industry, which explains why the last rows from $n+1$ to s are made of zeros, because final goods are not used as inputs. It also shows (in columns) the commodity composition of each industry's total inputs.

Matrix B is derived by dividing the use matrix U of dimensions ($s \times t$) by the diagonal matrix of industry total output $\widehat{g}$ of dimension ($t \times t$):

$$B = U\widehat{g}^{-1}, \tag{11}$$

$$
\begin{array}{cc}
\textit{Commodities} & \textit{Industries} \\
 & \begin{bmatrix} 1 & \cdots\cdots & l & l+1 & \cdots\cdots & q & q+1 & \cdots\cdots & t \end{bmatrix} \\
\begin{bmatrix} 1 \\ \cdots \\ k \\ k+1 \\ \cdots \\ n \\ n+1 \\ \cdots \\ s \end{bmatrix} &
\begin{bmatrix}
b_{11} & \cdots & b_{1l} & b_{1,l+1} & \cdots & b_{1q} & b_{1,q+1} & \cdots & b_{1t} \\
 & & & & \cdots & & & & \\
b_{k1} & \cdots & b_{kl} & b_{k,l+1} & \cdots & b_{kq} & b_{k,q+1} & \cdots & b_{kt} \\
0 & \cdots & 0 & b_{k+1,l+1} & \cdots & b_{k+1,q} & b_{k+1,q+1} & \cdots & b_{k+1,t} \\
 & & & & \cdots & & & & \\
0 & \cdots & 0 & b_{n,l+1} & \cdots & b_{nq} & b_{n,q+1} & \cdots & b_{nt} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 & & & & \cdots & & & & \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}
\end{array}. \tag{12}
$$

Table 3. A Make–Use Input–Output Table for Joint Production

	Commodities									*Industries*									*Total output*
Comm.	*1*	. . .	*k*	*k+1*	. . .	*n*	*n+1*	. . .	*s*	*1*	. . .	*l*	*l+1*	. . .	*q*	*q+1*	. . .	*t*	
										Use table: U									
1										B_{11}	. . .	B_{11}	$B_{1,\,l+1}$	. . .	B_{1q}	$B_{1,\,q+1}$	. . .	B_{1t}	
. . .										. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	
k										B_{k1}	. . .	B_{kl}	$B_{k,\,l+1}$	. . .	B_{kq}	$B_{k,\,q+1}$	. . .	B_{kt}	
k+1										0	. . .	0	$B_{k+1,\,l+1}$	. . .	$B_{k+1,\,q}$	$B_{k+1,\,q+1}$	. . .	$B_{k+1,\,t}$	
. . .										. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	*q*
n										0	. . .	0	$B_{k+1,\,l+1}$	. . .	B_{nq}	$B_{n,\,q+1}$	. . .	B_{nt}	
n+1										0	. . .	0	0	. . .	0	0	. . .	0	
. . .										. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	
s										0	. . .	0	0	. . .	0	0	. . .	0	
Indus.	*Make table: V*																		
1	D_{11}	. . .	D_{k1}	$D_{k+1,1}$	. . .	D_{n1}	0	. . .	0										
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .										
l	D_{1l}	. . .	D_{kl}	$D_{k+1,\,l}$	. . .	D_{nl}	0	. . .	0										
l+1	0	. . .	0	$D_{k+1,\,l+1}$	. . .	$D_{n,\,l+1}$	0	. . .	0										
. . .	. . .	. . .	. . .	. . .	. . .	. . .		. . .											*g*
q	0	. . .	0	$D_{k+1,\,q}$	. . .	D_{nq}	0	. . .	0										
q+1	0	. . .	0	$D_{k+1,\,q+1}$	. . .	$D_{n,\,q+1}$	$D_{n+1,\,q+1}$	. . .	$D_{s,\,q+1}$										
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .										
t	0	. . .	0	$D_{k+1,\,t}$	. . .	D_{nt}	$D_{n+1,\,t}$	. . .	D_{st}										
Total input					*q'*									*g'*					

In the case of joint production, one particular process can produce several different products, which in our demonstration are exclusive by-products, as previously defined, which have as such the same input structure. Even though in the real world pure or exclusive by-products are quite rare, and there are not so often more than two exclusive by-products for a particular joint production process, from a theoretical point of view we must consider a general case in which any process is able to produce a variety of different joint products, in the sense of exclusive by-products. This also means that the same product can be produced by different industries or processes. Indeed, and as demonstrated by Sraffa, we recall that this is necessary if the number of processes is ultimately “to be brought to equality with the number of commodities” so that prices (for Sraffa) or in our demonstration labor values, may be determined. In any case, we already explained why in the real world there is generally a multiplicity of processes which produce the same commodity, or in the present demonstration the same by-products. And such should be also the case in the theoretical world.

In order to give a mathematical representation of this reality, we will use below a “commodity-output-proportions” matrix D, which is an industry-by-commodity matrix. It is a matrix which has t rows (the industries) and s columns (the commodities), and is therefore also a rectangular matrix of dimensions ($t \times s$). It shows (in rows) the commodity composition of each industry’s total output of various commodities, which are exclusive by-products. It also shows (in columns) the share of each commodity’s total output that is produced by each industry.

This “commodity-output-proportions” matrix D, which shows the shares of each commodity’s total output that are produced by each industry, is obtained by dividing the make matrix V of dimensions ($t \times s$) by the diagonal matrix of commodity output $\widehat{q}$ of dimension ($s \times s$):

$$D = V\widehat{q}^{-1}, \tag{13}$$

$$
\begin{array}{cc}
 & \textit{Commodities} \\
\textit{Industries} & \begin{bmatrix} 1 & \cdots\cdots & k & k+1 & \cdots\cdots & n & n+1 & \cdots\cdots & s \end{bmatrix} \\
\begin{bmatrix} 1 \\ \cdots \\ l \\ l+1 \\ \cdots \\ q \\ q+1 \\ \cdots \\ t \end{bmatrix} &
\begin{bmatrix}
d_{11} & \cdots\cdots & d_{k1} & d_{k+1,1} & \cdots & d_{n1} & 0 & \cdots & 0 \\
 & & & & \cdots & & & & \\
d_{1l} & \cdots & d_{kl} & d_{k+1,l} & \cdots & d_{nl} & 0 & \cdots & 0 \\
0 & \cdots & 0 & d_{k+1,l+1} & \cdots & d_{n,l+1} & 0 & \cdots & 0 \\
 & & & & \cdots & & & & \\
0 & \cdots & 0 & d_{k+1,q} & \cdots & d_{nq} & 0 & \cdots & 0 \\
0 & \cdots & 0 & d_{k+1,q+1} & \cdots & d_{n,q+1} & d_{n+1,q+1} & \cdots & d_{s,q+1} \\
 & & & & \cdots & & & & \\
0 & \cdots & 0 & d_{k+1,t} & \cdots & d_{nt} & d_{n+1,t} & \cdots & d_{st}
\end{bmatrix}
\end{array}. \tag{14}
$$

On the basis of the industry technology assumption, a product j can be produced by a certain number of different industries k. Each industry k needs b_{ik} units of input i per unit of industry product j, where $b_{ik,}$ ($i = 1 \ldots n$) represents the industry technology of an industry k, and each industry k produces only a part of the total output of product j. This proportion of industry k in the total production of product j has a notation d_{kj}. So, all inputs i needed to produce 1 unit of product j by different industries can be written as follows:

$$a_{I,ij} = \sum_{k=1}^{n} b_{ik} d_{kj}, \text{ where I refers to the industry technology.} \tag{15}$$

The above formula shows that input i required to produce 1 unit of product j is a weighted average of the input structures of the industries where product j is produced: the weights are the proportions d_{kj} of each industry k in the total production of product j.

To make things clearer, let us use words to develop the above Equation (15) for coefficient a_{ij}, i.e., the share of the total production of commodity i which is used in the total production of commodity j:

$$a_{ij} = b_{i1}d_{j1} + b_{i2}d_{j2} + \ldots + b_{ij}d_{jj} + \ldots + b_{it}d_{jt}. \tag{16}$$

This means that a_{ij} is the sum of the share of total production of good i used in industry 1, multiplied by the share of industry 1 in the total production of good j, plus the share of total production of good i used in industry 2, multiplied by the share of industry 2 in the total production of good j . . . plus the share of total production of good i used in industry j, multiplied by the share of industry j in the total production of good j . . . plus the share of total production of good i used in industry t, multiplied by the share of industry t in the total production of good j.

Thus, through the additional information provided by the shares of various industries in the total production of a particular product (which could be defined as the "market shares" of these industries for this product), we are able to obtain the additional equations which allow for disentangling the initially hidden input structure of particular by-products. These additional equations in the form of Equation (16) can be put in matrix format and thus be written as:

$$A_{I(s,s)} = B_{(s,t)} D_{(t,s)}. \tag{17}$$

B is a commodity-by-industry rectangular matrix of dimensions ($s \times t$), and D as defined above is an industry-by-commodity rectangular matrix of dimensions ($t \times s$). Therefore, matrix A_I is a commodity-by-commodity matrix of dimensions ($s \times s$). Thus, A_I is the input–output coefficient matrix that describes commodities directly required to produce other commodities. It is a square matrix. For UN and US statisticians in the sources referred to above, as well as for Peter Flaschel in a 1983 article on "Actual Labor Values in a General Model of Production" (Flaschel 1983, 435–454), which uses their method, this square matrix A is invertible. This is a necessity for statisticians in order to calculate the commodity-by-commodity, total requirements matrix (ITC), which is derived as:

$$T = [I - A]^{-1} = [I - BD]^{-1}. \tag{18}$$

But it is also a necessity for theoreticians, who want to calculate values by using the usual and well-known formula:

$$V = VA + L, \tag{19}$$

which implies that $V = L\,(I - A)^{-1}$. (20)

One last difficulty deriving from the existence of joint production and which has to be addressed at this stage is indeed the nature of vector L, which appears in the above equation, because if we go back to our initial description of a joint production system in section 6.2 above, we note that the only information that we have regarding the quantities of labor time used in the system is the quantities by industry, which is logical because the nature of by-products does not allow us to apportion these quantities to the various joint products of a particular industry.

Therefore, the row vector of these quantities of labor has dimension of ($1 \times t$) and is:

$$V^I = \left(l_1^I + l_2^I + \ldots + l_t^I\right), \tag{21}$$

where index I stands for industry.

On this basis, if we wanted to calculate values, using matrix B, we would be unable to do it directly, because the unknown values to be determined would be both the values produced by each industry (vector V^I), and the values of each commodity, i.e., the elements of vector V^C:

$$V^I_{(1,t)} = V^C_{(1,s)} B_{(s,t)} + L^I_{(1,t)}. \tag{22}$$

Fortunately, there is a solution to this problem, which comes down to using the same tool as the one devised by statisticians, i.e., matrix D which gives us the commodity-output proportions. If indeed we multiply both terms of the above equation by matrix D, we obtain:

$$V^I_{(1,t)} D_{(t,s)} = V^C_{(1,s)} B_{(s,t)} D_{(t,s)} + L^I_{(1,t)} D_{(t,s)}. \tag{23}$$

As was already demonstrated, matrix D indeed provides a key for passing from industries to commodities, not only through the transformation of matrix B, but also for the transformation of vector V^I and L^I, since it allows us to write: $V^C = V^I D$ and $L^C = L^I D$, from which we derive:

$$V^C_{(1,s)} = V^C_{(1,s)} A_{(s,s)} + L^C_{(1,s)}. \tag{24}$$

Now that we have been able to overcome the difficulties linked to the existence of by-products, with the help of a method developed by statisticians and perfectly suited to the features of exclusive by-products, through the design of a matrix supposed to "behave well," the calculation of actual labor values seems to be quite simple. From a theoretical point of view, however, things cannot be as simple as they seem, and this is for several reasons.

The first reason for why things are not as simple as they might seem has to do with the basic data on which the method is based. Indeed, in his article just cited, Flaschel sticks fully to the SNA method, and since the quantities used by statisticians in their input–output tables are the monetary values of the commodities, he introduces prices in the calculation of the share of each industry in the production of joint products. This explains that the variables which appear in Flaschel's article are therefore what he calls the "relative sales values," from which are derived the monetary market shares of these commodities. Prices are therefore introduced in the calculation of values, a position that Flaschel fully endorses when he writes: "for our intents, labor values v were made to depend on prices p by the use of sales values C_{kj}" (Flaschel 1983, 453).

However, such a position is wrong for several reasons. First, because to know the market shares implies knowing which quantities have been sold on the market, and at which prices. But when we stand at the theoretical level of production and values, there are no prices, because as we already mentioned previously, values are logically and chronologically anterior to the sale of commodities at monetary prices, which appear only after production has taken place, and when this production is being sold on the market.

A second reason has also been emphasized by Toker, and has to do with the distinction between average values and individual values of commodities, which necessarily appear when the same commodity can be produced by several methods. This author indeed writes:

> [U]nlike single-production systems, in joint production systems prices at zero rate of profit are still prices *par excellence* but not labor values; in the latter perfect competition can ensure, at any rate of profit (including zero) the uniqueness of prices, but not the uniqueness of labor values. (Toker 1984, 152)

A third and last reason why Flaschel should not have based his calculation of labor values on monetary market shares is that values and prices do not have the same dimension: values have the dimension of time, which is the standard of values, and prices are pure dimensionless numbers, or scalars. Therefore, they cannot be put at the same level, which means that values are not a special form of prices which would appear when the rate of profit is zero.

In his highly cited book *Marx after Sraffa*, Steedman (1977) considers that the rate of profit, the prices of production, and the social allocation of labor power can all be determined without any reference to value magnitudes because all these economic variables can be calculated in the framework of Sraffa's production prices, values being just a particular case of these prices when the profit rate is zero. Obviously, this view, also known as reverse transformation, is totally wrong, if only because as shown above Sraffa's system of production prices is incapable of accommodating joint production. But more fundamentally because both systems have incompatible standards: the dimension of values is by definition labor time, whereas in Sraffa's theory the standard of measurement is a basket of basic commodities, with both standards belonging to different dimensions, and no key for passing from one system to the other. Indeed, with $r = 0$, when you multiply the real wage w/l in Sraffa's system by l_i to calculate a so-called value for commodity i, labor time is ipso facto eliminated, and what you obtain is only a part of a basket of commodities.

Moreover, the use of monetary market shares is not needed as a mathematical necessity for the determination of values in the case of joint production, since matrix D above can perfectly be defined in terms of ratios between the quantities of a commodity produced in various industries and the total quantity of this same commodity produced by the whole production system: it is a matrix of "commodity-output proportions." This is indeed what has been done above by writing: $D = V\hat{q}^{-1}$, with the primary elements of V being mere quantities.

There is also a last reason for why things are not as simple as they might seem, which has to do with the existence of several distinct categories of commodities. Indeed, as soon as we introduce, as it is the case in this section, a distinction between final commodities and intermediate commodities, which is indispensable in a theory where production is seen as a transformation of the latter into the former, things become more complicated than they are in the statistical world where this distinction is not taken into account. The final goods are not transformed into intermediate goods, and there are two types of these last ones, which has some consequences on the nature of matrices involved in the representation of production.

This implies that matrix A, as the product of both matrices B and D, which themselves are as exposed above in Equations (12) and (14), presents itself in the following form, with matrices B and D decomposed into submatrices:

$$A = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} D_{11} & D_{12} & 0 \\ 0 & D_{22} & 0 \\ 0 & D_{32} & D_{33} \end{bmatrix} = \begin{bmatrix} B_{11}D_{11} & [B_{11}D_{12}+B_{12}D_{22}+B_{13}D_{32}] & B_{13}D_{33} \\ 0 & [B_{22}D_{22}+B_{23}D_{32}] & B_{23}D_{33} \\ 0 & 0 & 0 \end{bmatrix}. \quad (25)$$

It is obvious that the resulting matrix A is singular and therefore not invertible, because matrix B is itself a singular matrix: its $s - n$ last rows are zeros, which simply reflects the fact that final goods by definition do not enter into the production of any good. By getting rid of these last rows, we can nevertheless rewrite matrix A as a new matrix A^*, which can be written as:

$$A^* = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \end{bmatrix}. \quad (26)$$

It is clear that A^* is a rectangular matrix of dimensions $(n \times s)$, and therefore is not invertible.

As for the dimensions of the six submatrices contained in matrix A^*, they are the following:

A_{11} is a square matrix with k rows and k columns, of dimensions $(k \times k)$

A_{12} is a rectangular matrix with k rows and $n - k$ columns, of dimensions $(k \times n - k)$

A_{13} is a rectangular matrix with k rows and $s - n$ columns, of dimensions $(k \times s - n)$

A_{21} is a zero rectangular matrix, with $n - k$ rows and k columns, of dimensions $(n - k \times k)$

A_{22} is a square matrix with $n - k$ rows and $n - k$ columns, of dimensions $(n - k \times n - k)$

A_{33} a rectangular matrix with $n - k$ rows and $s - n$ columns, of dimensions $(n - k \times s - n)$

An observation which can be made from this matrix A^* is that we can isolate from it another submatrix $A^\#$, which is composed of the first two rows and first two columns of submatrices of matrix A^*, such as:

$$A^\# = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}. \quad (27)$$

In fact, submatrix A_{11} gives us the technical coefficients that correspond to the share of intermediate goods of the first type that enter in their own production, submatrix A_{12} those that correspond to the share of these same intermediate goods of the first type that enter in the production of intermediate goods of the second type, and matrix A_{22} those that correspond to the share of these last goods that enter only in their own production.

To be sure $A^{\#}$ is a square matrix of dimensions $n \times n$, but its columns are not linearly independent, which implies that $A^{\#}$ is a singular matrix and therefore is not invertible. Like what could be done for the calculation of values in the absence of joint production, we must therefore solve the problem of value determination by using a step-by-step approach. Indeed, once the system is split into three subsystems, it becomes possible to represent it in the following matrix format, where we have three matrix equations:

$$V_1 = V_1 A_{11} + L_1, \tag{28}$$

$$V_2 = V_1 A_{12} + V_2 A_{22} + L_2, \tag{29}$$

$$V_3 = V_1 A_{13} + V_2 A_{23} + L_3. \tag{30}$$

Only the first one is an independent matrix equation, in which A_{11} is a $k \times k$ matrix with entries a_{ij}. Such a matrix corresponds to what is usually called an open Leontief model, i.e., to an economy where there is some external source of demand for each industry: here this demand corresponds to the share of the production of intermediate commodities of the first type which is needed for the production of other intermediate and final commodities.

Matrix A_{11} is positive ($A_{11} > 0$), because $a_{ij} \geq 0$ for all i and j and $a_{ij} > 0$ for some i and j. We know also that A_{11} has row sums not exceeding 1, meaning that: $\sum_j a_{ij} \leq 1$ for all i.

Such a matrix is called sub-stochastic (it would be stochastic if each row sum were 1). In such a case, the demonstration has been made by Peterson and Olinick (1982) that $(I - A_{11})^{-1} > 0$. To be sure, they make the demonstration for matrices where column sums (and not row sums) are not exceeding 1, because they use the transposed matrix of matrix A_{11}, but this does not change either the demonstration or its outcome.

Let us first recall that by definition, matrix A is productive if there is a non-negative vector X such that $X \geq AX$. Their demonstration is then based on their theorem 5.1 (Peterson and Olinick 1982, 221–239):

Theorem 5.1: A sub-stochastic matrix A is productive if and only if $I - A$ is non-singular.

From theorem 5.1, matrix $I - A$ is invertible and necessarily therefore $(I - A_{11})^{-1} > 0$. The solution vector is $X = (I - A_{11})^{-1}B$.

This ensures that values are always positive, which allows us to write:

$$V_1 = \left(I - A_{11}\right)^{-1} L_1. \tag{31}$$

Then, knowing V_1, the vector of values of intermediate commodities of the first type, from Equation (29) we can get V_2, i.e., the vector of values of intermediate commodities of the second type, because A_{22} is a square matrix of dimensions $n - k$ by $n - k$, which allows for solving the second equation:

$$V_2 = V_1 A_{12} + V_2 A_{22} + L_2 \tag{32}$$

by writing:

$$V_2 = \left(I - A_{22}\right)^{-1} \left(V_1 A_{12} + L_2\right). \tag{33}$$

Finally, knowing V_1 and V_2, from Equations (31) and (33) we obtain V_3, the vector of values of final commodities, without any other additional calculation than displayed by Equation (30):

$$V_3 = V_1 A_{13} + V_2 A_{23} + L_3. \tag{34}$$

7. Conclusion

We have thus demonstrated that values can be calculated in a coherent way in the case of joint production. This means that we have arrived now at the conclusion of this article on joint production, which has shown that contrary to Sraffa's and Morishima's assertions it is perfectly possible to introduce joint production into a theoretical system without having negative labor values. In fact, such peculiarities appear in Sraffa's theory because of the need for him to build a standard commodity, which is quite a strange device in his system, and in Morishima's theory because of the introduction of other strange assumptions like the existence of non-productive sectors that use some inputs in larger quantities than the produced quantities of these same inputs.

This conclusion according to which there is no such thing as negative quantities of labor has the merit of being in line with both empirical reality and intuition, because in the real world, where indeed joint production does exist, albeit in small proportions compared to overall production, this particularity is never associated to negative values or prices, being understood that, as Flaschel mentions it rightly: "the labor value of a jointly produced free good should be zero" (Flaschel 1983, 449).

Although we disagree with Flaschel on a particular point, i.e., the need to use prices in the calculation of values, we cannot but adhere to his conclusion which confirms the relevance of the labor theory of value and the existence of "actual labor values in a general model of production." What this article has shown indeed is that as long as it stays confined to the theoretical sphere of production and is not unduly extended to that of exchange, Marx's labor theory of value is perfectly compatible with joint production. On the basis of this finding, we can state a general principle: "Joint production, defined as the production of several distinct commodities by a single method of production, is compatible with the existence and calculation of positive labor values, as long as different methods produce these commodities in different proportions."

Notes

1. An industry is generally identified by broad categories of products, such as: construction industry, chemical industry, petroleum industry, automotive industry, electronic industry, meatpacking industry, and so on.
2. Since we are at the stage of production, commodities have not yet been put on the market. Therefore, it would be preferable to refer to the output proportions of the commodity in question, even though these proportions should be based on the market shares of the previous production period.

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